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Vibration Theory, Vol. 1A

linear vibration theory solved problems, 3rd ed.

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Publication date:
2004

Document Version
Early version, also known as pre-print

[Link to publication from Aalborg University](#)

Citation for published version (APA):

Nielsen, S. R. K. (2004). *Vibration Theory, Vol. 1A: linear vibration theory solved problems, 3rd ed.* (3rd edition ed.) Department of Civil Engineering, Aalborg University. U/ Vol. 1A No. 2004-2

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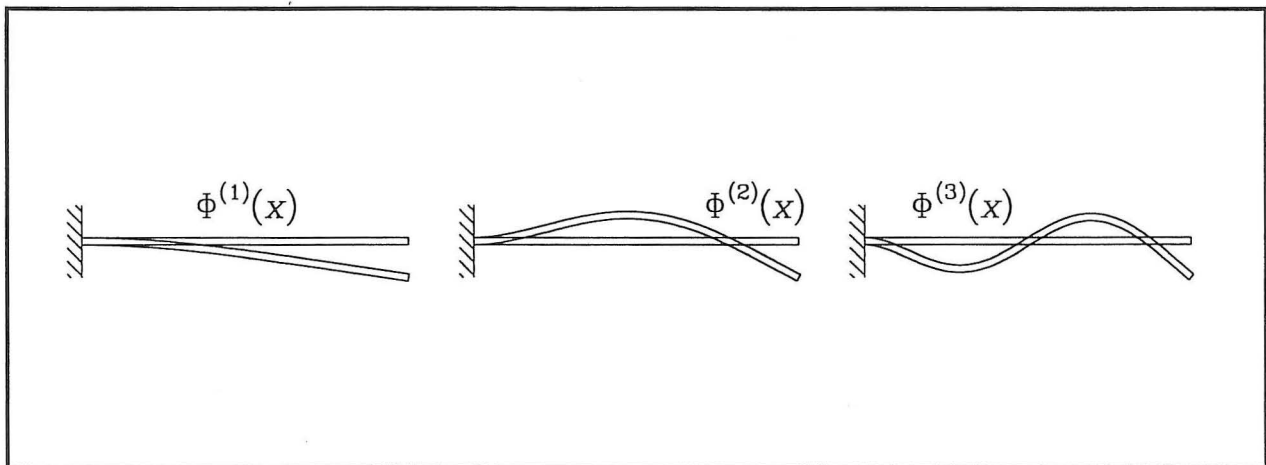
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VIBRATION THEORY, VOL. 1A

Linear Vibration Theory Solved Problems, 3rd ed.

Søren R. K. Nielsen



Aalborg tekniske Universitetsforlag
June 2004

PREFACE

The present collection of solved problems has been published as a supplement to the textbook *Vibration Theory, Vol. 1. Linear Vibration Theory*, Aalborg tekniske Universitetsforlag, 1998, which is used in the introductory course on structural dynamics on the 8th semester at Aalborg University for M.Sc. students in structural engineering. Throughout the text, references are made to the above textbook with the format "(3-137)". For all examination problems the average score among the participating students has been indicated by the format " $\mu = 18.8\%$ ".

Kim J. Mørk, Ph.D. has formulated the problems for the examination on October 3, 1990 and John Asmussen, Ph.D. has formulated the problems for the examinations on June 21, 1996 and August 26, 1996. Problem 1, September 14, 1988, was originally given for the final year examination on structural dynamics 1967-1968 at the Technical University of Denmark, and has been reprinted in this collection by courtesy of Professor, dr. techn. C. Dyrbye. However, I am the only one to blame for the translation into English, as well as for the indicated solutions.

The present 2nd edition of the book appears as an update of the 1st edition published November, 1993. Problems on the analytical determination of eigenfrequencies of plane frames have been skipped in the present edition, since this subject is no longer taught in the course. Instead, problems on dynamic modelling based on the finite element method have been included. Further, all problems on linear stochastic vibration theory included in the 1st edition have also been omitted.

Mrs. Solveig Hesselvang and Mrs. Kirsten Aakjær have skillfully typed the manuscript, and Mrs. Norma Hornung has prepared the drawings. The help from all of them is gratefully acknowledged.

Aalborg University, September 1998

Søren R. K. Nielsen

The present 3rd edition of my textbook on linear vibration theory is unchanged in comparison to the 2nd edition. Only discovered typing errors have been corrected.

Aalborg University, July 2004

Søren R. K. Nielsen

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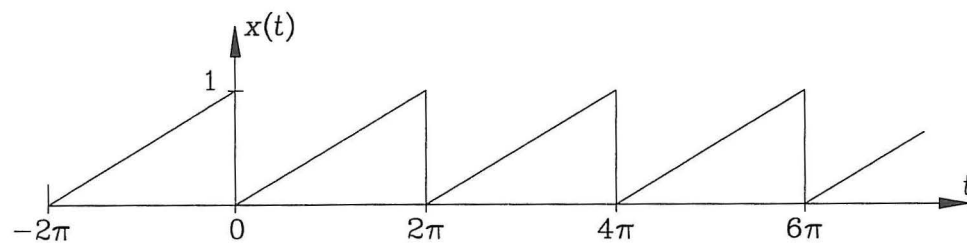
1. EXERCISES

1.1 Lecture 1

SUPPLEMENTARY PROBLEMS

- September 5, 1990. Problem 1.
- October 3, 1990. Problem 1, question 3.
- August 24, 1994. Problem 1.
- June 21, 1996. Problem 1.

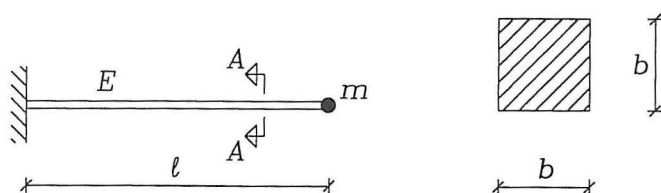
PROBLEM 1



Determine the Fourier series for the saw-toothed wave shown in the figure. The result is wanted in the form of the exponential series

$$x(t) = \sum_{m=-\infty}^{\infty} A_m e^{i\omega_m t}$$

PROBLEM 2

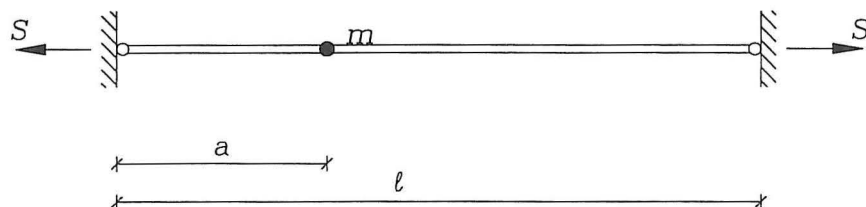


The beam in the figure is a homogeneously, massless Bernoulli-Euler beam of the length l , the height h , the width b and the elasticity modulus E . At the free end a point

mass m is attached. Determine the circular eigenfrequency, the eigenfrequency and the vibration period for the system.

Data: $m = 5$ kg, $l = 0.25$ m, $b = 0.01$ m, $h = 0.01$ m, $E = 2.0 \cdot 10^{11}$ Pa.

PROBLEM 3



The figure shows a massless rope of the length l , which is prestressed with the force S . At a distance a from one of the ends a point mass m is attached. Only small vibrations normal to the equilibrium state are considered, and the horizontal component of the rope force is assumed to be constant equal to S during vibrations. Determine the circular eigenfrequency of the system.

SOLUTIONS

PROBLEM 1

$$T = 2\pi \Rightarrow \omega_1 = \frac{2\pi}{T} = 1 \Rightarrow$$

$$\omega_m t = m\omega_1 t = mt \tag{1}$$

For $t \in [0, 2\pi[$ the signal is given as

$$x(t) = \frac{t}{2\pi} \tag{2}$$

From (A-3) it then follows that

$$a_m = \frac{2}{2\pi} \int_0^{2\pi} \frac{t}{2\pi} \cos(mt) dt = \begin{cases} 1 & , m = 0 \\ 0 & , m = 1, 2, \dots \end{cases} \tag{3}$$

$$b_m = \frac{2}{2\pi} \int_0^{2\pi} \frac{t}{2\pi} \sin(mt) dt = -\frac{1}{m\pi} \quad , \quad m = 1, 2, \dots \quad (4)$$

From (A-4) and (A-5) it follows that

$$x(t) = \frac{1}{2} + \sum_{m=1}^{\infty} \left(-\frac{1}{m\pi} \right) \sin(mt) = \sum_{m=-\infty}^{\infty} A_m e^{imt} \quad (5)$$

where

$$A_m = \begin{cases} \frac{1}{2} \left(0 - i \left(-\frac{1}{m\pi} \right) \right) & , \quad m > 0 \\ \frac{1}{2} & , \quad m = 0 \\ \frac{1}{2} \left(0 + i \left(-\frac{1}{(-m)\pi} \right) \right) & , \quad m < 0 \end{cases} = \begin{cases} \frac{1}{2} & , \quad m = 0 \\ \frac{i}{2m\pi} & , \quad m \neq 0 \end{cases} \quad (6)$$

Notice, at the discontinuity points at $t = n2\pi$, $n = 0, \pm 1, \pm 2, \dots$ the series (5) is seen to give the value $x(t) = \frac{1}{2}$ in accordance with (A-1).

PROBLEM 2

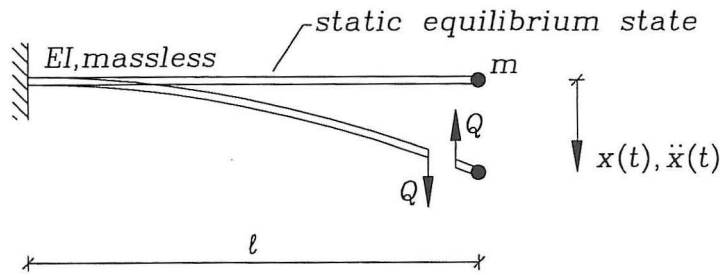


Fig. 1: Free mass during vibrations

The beam is massless. Hence, the system has a single degree of freedom. The mass m is cut free, and the shear force Q is applied as an external load to the mass. Newton's 2nd law of motion provides

$$m\ddot{x} = -Q \quad (1)$$

From static analysis of the linear elastic beam to the left of the cut it follows that

$$Q = kx \quad , \quad k = 3 \frac{EI}{l^3} \quad (2)$$

From (1) and (2) the equation of motion is then obtained

$$m\ddot{x} + kx = 0 \Rightarrow$$

$$\omega_0^2 = \frac{k}{m} = 3 \frac{EI}{ml^3} \quad (3)$$

$$I = \frac{1}{12}bh^3 = \frac{1}{12} \cdot 0.01 \cdot 0.01^3 = \frac{1}{12} \cdot 10^{-8} \text{m}^4 \quad (4)$$

$$\omega_0 = \sqrt{\frac{3 \cdot 2.0 \cdot 10^{11} \cdot \frac{1}{12} \cdot 10^{-8}}{5 \cdot 0.25^3}} \frac{\text{rad}}{\text{s}} = 80 \frac{\text{rad}}{\text{s}} \quad (5)$$

$$f_0 = \frac{\omega_0}{2\pi} = 12.73 \text{ Hz} \quad (6)$$

$$T_0 = \frac{1}{f_0} = 0.0785 \text{ s} \quad (7)$$

PROBLEM 3

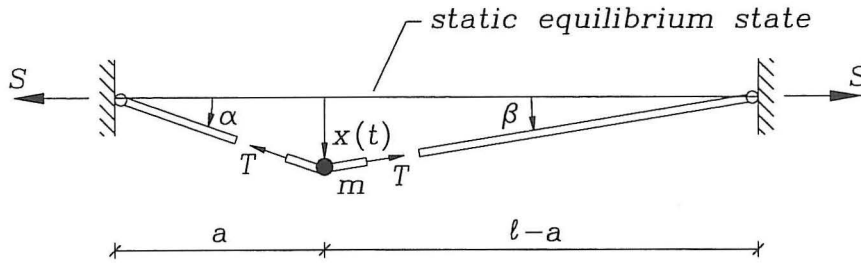


Fig. 1: Free mass during vibrations.

The system has a single degree of freedom $x(t)$, which is chosen as the vertical displacement from the static equilibrium state with a sign as shown in fig. 1. During vibrations the force in the rope is changed from S to T . If $x \ll \min(a, l-a)$, T and S are almost of equal magnitude, i.e. $T \simeq S$. However, the direction of T has changed, and the vertical component of the rope force influences the motion of the particle. The mass is cut free, and the vertical components of T are applied as external loads to the mass. Newton's 2nd law of motion provides

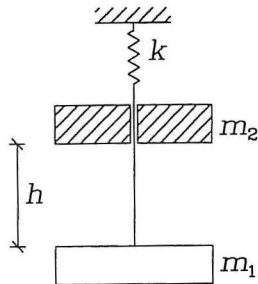
$$\begin{aligned} m\ddot{x} &= -T \sin \alpha - T \sin \beta \simeq -S \frac{x}{a} - S \frac{x}{l-a} \Rightarrow \\ m\ddot{x} + \left(\frac{1}{a} + \frac{1}{l-a} \right) Sx &= 0 \end{aligned} \quad (1)$$

The circular eigenfrequency then follows from (2-7)

$$\omega_0 = \sqrt{\frac{l}{a(l-a)} \frac{S}{m}} \quad (2)$$

1.2 Lecture 2

PROBLEM 1

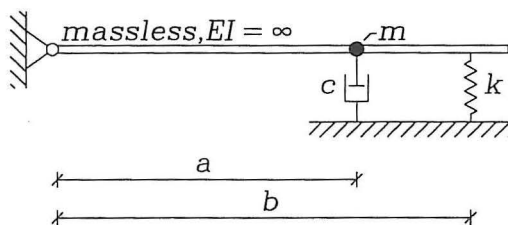


A mass m_1 is suspended in a linear elastic spring with spring constant k . A second mass in rest performs a free fall of the height h , and fix to the mass m_1 in a perfectly inelastic impact. Determine the succeeding motion of the system.

PROBLEM 2

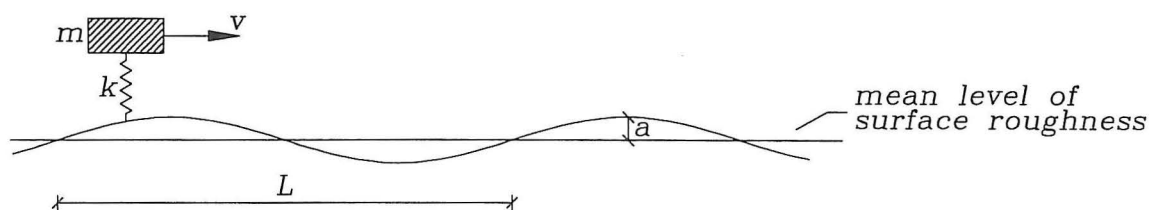
Determine the motion of a linear viscous damped system for the initial values $x_0 = 0$, $\dot{x}_0 = v_0$. Plot the non-dimensional response $\frac{\omega_0 x(t)}{v_0}$ versus the non-dimensional time $\omega_0 t$ for the cases $\zeta = 2.0$, $\zeta = 1.0$, $\zeta = 0.5$.

PROBLEM 3



The beam in the figure is infinitely stiff and massless. Determine the equation of motion, the circular eigenfrequency and the damping ratio of the system. Determine the critical value c_k of the damping constant c .

PROBLEM 4



The highly idealized vehicle shown in the figure consists of a mass m and a linear elastic spring with the spring constant k in permanent contact with the surface of a rough road. The profile of the road is approximated with a sine wave with the amplitude a and the wave length L . The vehicle is assumed to move at the constant speed v . Determine the motion of the vehicle in the vertical direction, and determine the most unfavourable speed of the vehicle.

SOLUTIONS

PROBLEM 1

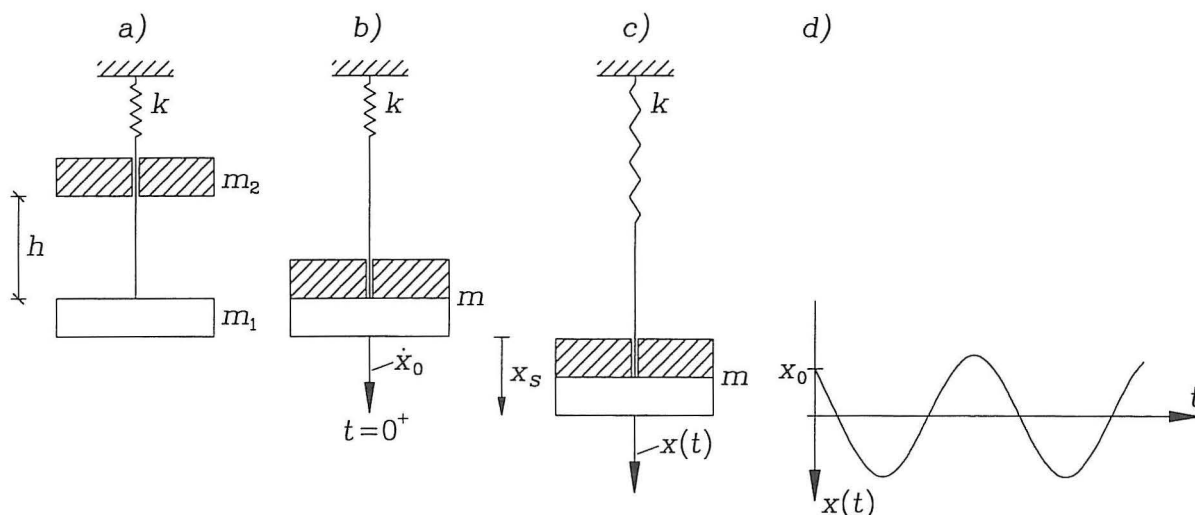


Fig. 1: a) Mass m_2 at rest before free fall. b) Fixation of masses. c) New static equilibrium state. d) Vibrations from new static equilibrium state.

After the mass m_2 is fixed to the mass m_1 at the time $t = 0^+$ a new static equilibrium

state is attained. The elongation of the spring x_s from the old to the new equilibrium state is determined from (see fig. 1c)

$$m_2 g = k x_s \Rightarrow$$

$$x_s = \frac{m_2 g}{k} \quad (1)$$

g is the acceleration of gravity. After the fixation of the mass m_2 the system has a single degree of freedom $x(t)$, which is measured from the new static equilibrium state as shown in fig. 1c. Then the equation of motion for the new system mass $m = m_1 + m_2$ is given as

$$m \ddot{x} + kx = 0 \quad , \quad m = m_1 + m_2 \Rightarrow$$

$$x(t) = \frac{\dot{x}_0}{\omega_0} \sin(\omega_0 t) + x_0 \cos(\omega_0 t) \quad , \quad t \geq 0 \quad (2)$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{m_1 + m_2}} \quad (3)$$

The initial displacement x_0 is given as

$$x_0 = -x_s = -\frac{m_2 g}{k} \quad (4)$$

Ignoring the air friction the velocity of the mass m_2 at the time $t = 0^-$ is given as $v = \sqrt{2gh}$. The initial velocity \dot{x}_0 of the mass m can then be determined from the equation of momentum

$$m_2 v = m \dot{x}_0 \Rightarrow$$

$$\dot{x}_0 = \frac{m_2}{m_1 + m_2} \sqrt{2gh} \quad (5)$$

Inserting (3), (4) and (5) into (2) provides

$$x(t) = \frac{m_2}{m_1 + m_2} \sqrt{2gh} \sqrt{\frac{m_1 + m_2}{k}} \sin(\omega_0 t) - \frac{m_2 g}{k} \cos(\omega_0 t) \quad , \quad t \geq 0 \quad (6)$$

PROBLEM 2

$(x_0, \dot{x}_0) = (0, v_0)$ and $\zeta = 2.0$ in (2-42) provide

$$\begin{aligned} x(t) &= \frac{v_0}{\omega_0 \sqrt{3}} \exp(-2\omega_0 t) \sinh(\sqrt{3}\omega_0 t) \Rightarrow \\ \frac{\omega_0 x(t)}{v_0} &= \frac{1}{\sqrt{3}} \exp(-2\omega_0 t) \sinh(\sqrt{3}\omega_0 t) \end{aligned} \quad (1)$$

$(x_0, \dot{x}_0) = (0, v_0)$ and $\zeta = 1.0$ in (2-41) provide

$$\begin{aligned} x(t) &= v_0 t \exp(-\omega_0 t) \Rightarrow \\ \frac{\omega_0 x(t)}{v_0} &= \omega_0 t \exp(-\omega_0 t) \end{aligned} \quad (2)$$

$(x_0, \dot{x}_0) = (0, v_0)$ and $\zeta = 0.5$ in (2-40) provide

$$\begin{aligned} x(t) &= \frac{v_0}{\omega_0 \sqrt{0.75}} \exp(-0.5\omega_0 t) \sin(\sqrt{0.75}\omega_0 t) \Rightarrow \\ \frac{\omega_0 x(t)}{v_0} &= \frac{1}{\sqrt{0.75}} \exp(-0.5\omega_0 t) \sin(\sqrt{0.75}\omega_0 t) \end{aligned} \quad (3)$$

Below, (1), (2) and (3) have been plotted as a function of $\omega_0 t$.

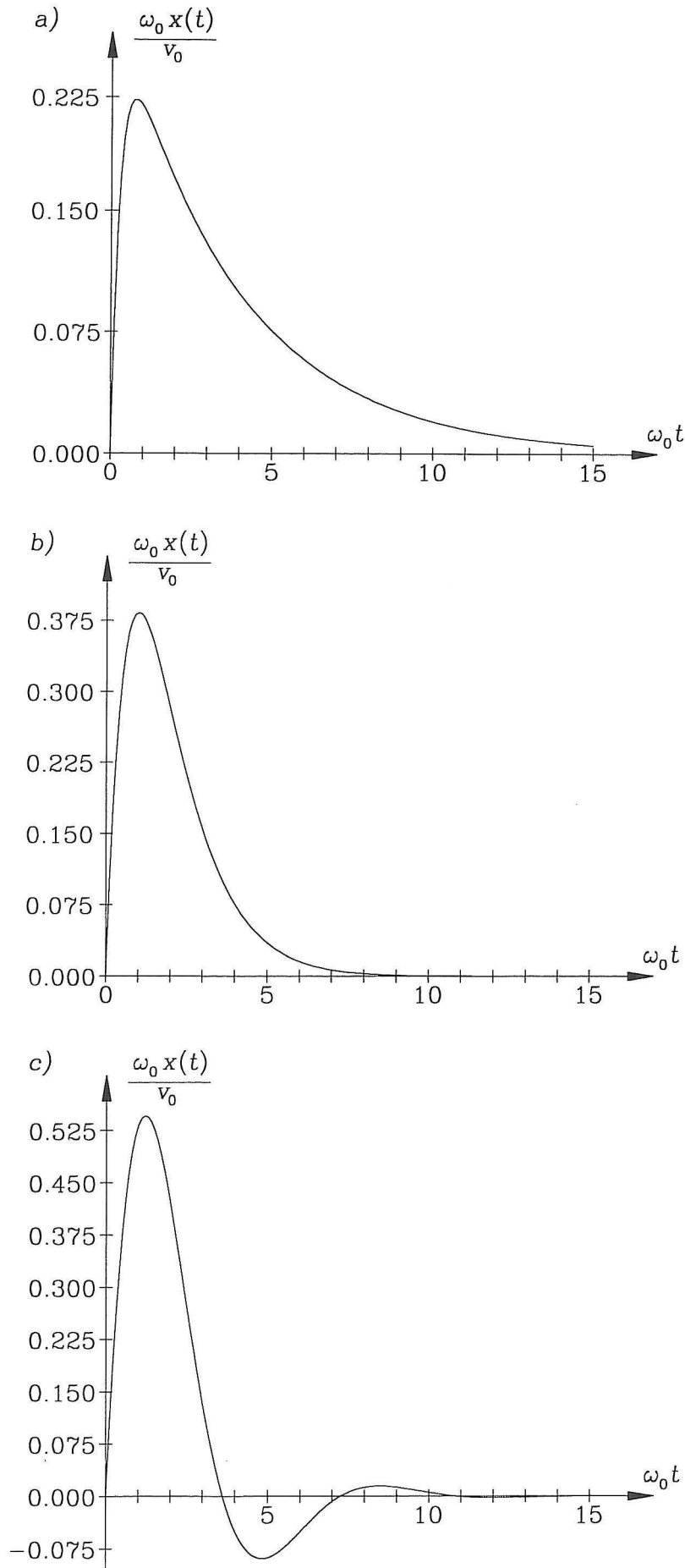


Fig. 1: Eigenvibrations. a) Overcritical damping, $\zeta = 2.0$. b) Critical damping, $\zeta = 1.0$. c) Undercritical damping, $\zeta = 0.5$.

PROBLEM 3

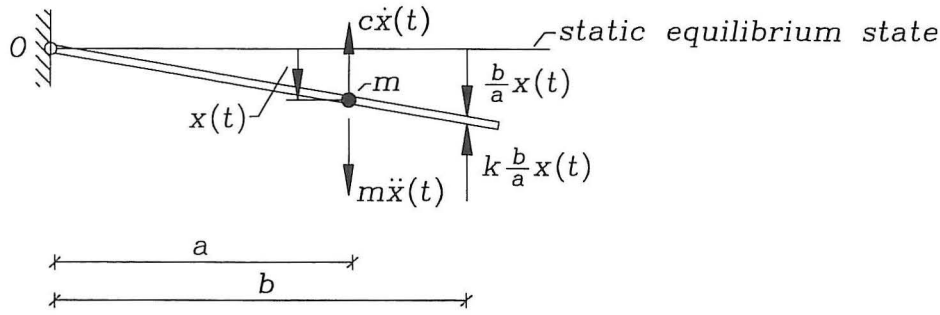


Fig. 1: Forces on free beam.

The system has a single degree of freedom, which is selected as the vertical displacement $x(t)$ of the mass from the static equilibrium state. The beam is cut free from the spring and the damper, and the spring force $k \cdot \frac{b}{a} x(t)$ and the damper force $c\dot{x}(t)$ are applied as external forces with signs as shown in fig. 1.

The equation of motion of the system can be formulated by means of the equation of moment of momentum applied at node O. Then the unknown reaction force at point O does not contribute to the external force moment. Further, the static gravity forces are eliminated from the equation of motion, because the vibrations are measured from the static equilibrium state. Assuming small vibrations from the static equilibrium state it then follows that

$$\frac{d}{dt}(m\dot{x}a) = -c\dot{x} \cdot a - k \frac{b}{a} x \cdot b \quad (1)$$

The momentum of the mass is $m\dot{x}$, and its moment of momentum around O is $m\dot{x}a$. The left-hand side represents the time derivative of the moment of momentum. According to the equation of moment of momentum this quantity is equal to the moment of all external forces around O in the same direction, as indicated on the right-hand side of (1).

From (1) it follows that

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0 \quad (2)$$

$$\omega_0^2 = \frac{k}{m} \frac{b^2}{a^2} \Rightarrow$$

$$\omega_0 = \frac{b}{a} \sqrt{\frac{k}{m}} \quad (3)$$

The static equilibrium state is defined as the position of the mass, when the vehicle is at rest at a position, where $y(s) = 0$, i.e. at the mean level of the road profile. The displacement $x(t)$ of the vehicle mass is measured from this equilibrium state with a sign as shown in fig. 1.

At the time t the moving vehicle is at the position s with the surface roughness $y(s)$. The displacement is $x(t)$. Hence, the elongation of the spring is $x(t) - y(s)$. $t = 0$ is selected at the instant of time, where the vehicle is at the position $s = 0$. Then $s = vt$, see fig. 1. The mass is cut free, and the spring force $k(x(t) - y(vt))$ is applied as external force with signs as shown in fig. 1. Newton's 2nd law of motion for the free vehicle mass then provides

$$m\ddot{x} = -k(x(t) - y(vt)) \Rightarrow$$

$$\ddot{x} + \omega_0^2 x(t) = \omega_0^2 y(vt) \quad (2)$$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad (3)$$

ω_0 is the circular eigenfrequency of the vehicle. Inserting (1) on the right-hand side of (2) provides

$$\ddot{x} + \omega_0^2 x = \omega_0^2 a \sin(\omega t) \quad (4)$$

$$\omega = 2\pi \frac{v}{L} \quad (5)$$

ω as given by (5) is an artificial circular excitation frequency for the problem. The vehicle is assumed to have been driving in a sufficiently long time that any response from the initial values has been dissipated. Then only the stationary motion of (4) remains. This becomes

$$x(t) = \frac{\omega_0^2}{\omega_0^2 - \omega^2} a \sin(\omega t) \quad (6)$$

The most unfavourable speed v_c is the one causing the resonance, as occurs if $\omega = \omega_0$. In this case

$$2\pi \frac{v_c}{L} = \frac{2\pi}{T_0} \Rightarrow$$

$$v_c = \frac{L}{T_0} \quad (7)$$

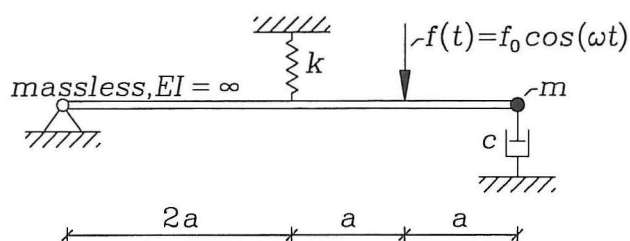
T_0 is the eigenperiod of the oscillator. In case $v = v_c$, the amplitudes of the oscillator increase infinitely proportional with the time. Physically, the critical velocity is attained, when the vehicle is travelling one wave length L during one eigenperiod T_0 .

1.3 Lecture 3

SUPPLEMENTARY PROBLEMS

- May 14, 1987. Problem 3, question 1.
- June 26, 1987. Problem 3, question 1.
- June 16, 1989. Problem 2, question 3 (only the formulation of the equation of motion).
- October 3, 1990. Problem 1, questions 1 and 2.
- June 16, 1993. Problem 1.
- August 31, 1993. Problem 1.
- June 22, 1994. Problem 1.
- June 28, 1995. Problem 1.

PROBLEM 1

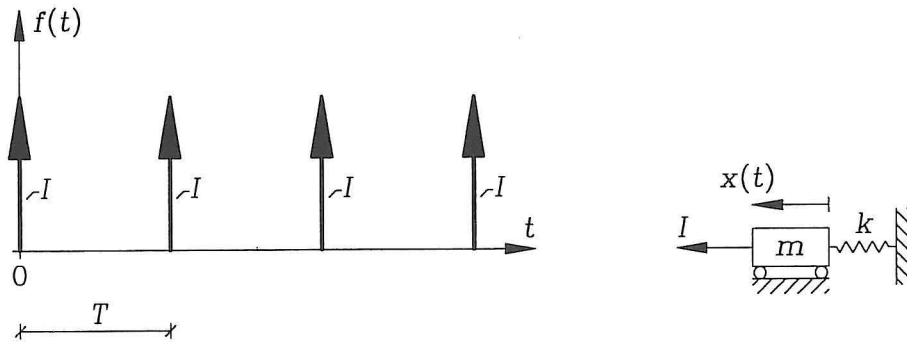


The beam in the figure is massless and infinitely stiff. The spring is linearly elastic with the spring constant k , and the damper is linear viscous with the damping constant c . The indirectly acting force $f(t)$ is harmonic and has been acting during exceedingly long time. Determine the stationary motion of the point mass m . Only small vibrations are considered.

PROBLEM 2

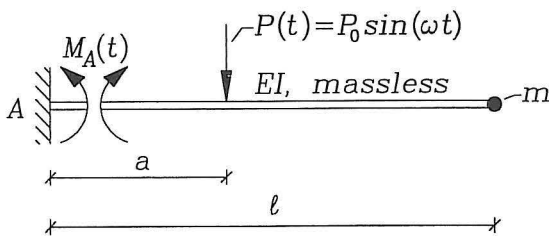
A linear viscous damped single degree-of-freedom system is subjected to the harmonic force $f(t) = f_0 \cos \omega t$. At the resonance the stationary amplitude is measured to be 0.58 cm. At the frequency ratio $\beta = \frac{\omega}{\omega_0} = 0.8$ the stationary amplitude becomes 0.46 cm. Determine the damping ratio of the system.

PROBLEM 3



An undamped single degree-of-freedom system is subjected to a periodic system of positive impulses I of negligible time duration. The period between the impulses is T . Determine the stationary motion of the system. Plot the displacement and the velocity of the response versus the non-dimensional time $\frac{t}{T}$ for $\omega_0 T = \frac{\pi}{2}$ and $\omega_0 T = \frac{5\pi}{2}$, where ω_0 is the undamped circular eigenfrequency of the system.

PROBLEM 4



The beam in the figure is a fixed-free homogeneously, massless Bernoulli-Euler beam free of damping. The length is l and the bending stiffness is EI . At the free end of the beam a point mass m is attached. The structure is at rest at the time $t = 0$, where the indirectly acting harmonic force $P(t) = P_0 \sin(\omega t)$ at the distance a from the fixation is applied. Determine the displacement response of the mass and the time variation of the bending moment $M_A(t)$ at the fixation at point A for $\omega = \omega_0$, where ω_0 is the circular eigenfrequency of the system.

SOLUTIONS

PROBLEM 1

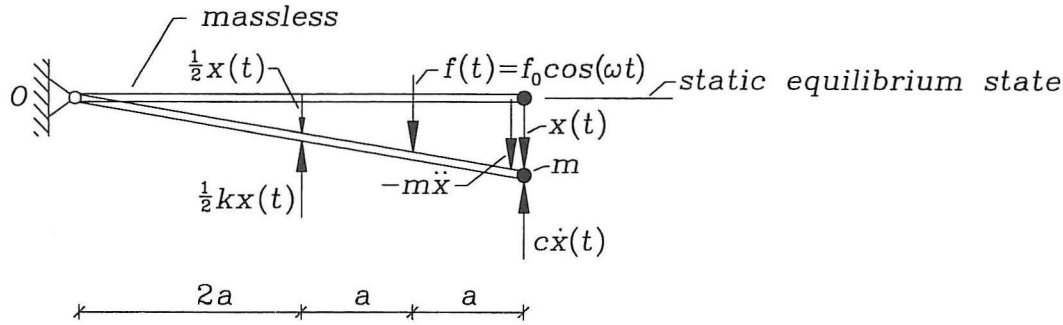


Fig. 1: Forces on free beam.

Since the beam is massless, the system has but a single degree of freedom, which is selected as the vertical displacement $x(t)$ of the mass from the static equilibrium state. The beam is cut free from the spring and the damper, and the spring force $\frac{1}{2}kx(t)$ and the damper force $c\dot{x}(t)$ are applied as external forces with signs as shown in fig. 1. Further, the inertial load $-m\ddot{x}$ is applied as an external load acting on the mass. According to d'Alembert's principle the equation of motion can then be formulated, expressing the static moment equilibrium around point O. Hence

$$(-m\ddot{x} - c\dot{x})4a + f(t) \cdot 3a - \frac{1}{2}kx \cdot 2a = 0 \Rightarrow$$

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2x = \frac{3}{4}\frac{f_0}{m}\cos(\omega t) \quad (1)$$

$$\omega_0^2 = \frac{k}{4m} \Rightarrow$$

$$\omega_0 = \frac{1}{2}\sqrt{\frac{k}{m}} \quad (2)$$

$$\zeta = \frac{c}{2m\omega_0} = \frac{c}{\sqrt{mk}} \quad (3)$$

The stationary solution of (1) reads, cf. (2-64), (2-69), (2-70)

$$x(t) = |X|\cos(\omega t - \Psi) \quad (4)$$

$$|X| = \frac{\frac{3}{4}f_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\zeta^2\omega_0^2\omega^2}} \quad (5)$$

$$\tan \Psi = \frac{2\zeta\omega_0\omega}{\omega_0^2 - \omega^2} \quad (6)$$

PROBLEM 2

The dynamic amplification factor is given as, see (2-72), (2-73)

$$\frac{|X|k}{|F|} = \frac{1}{\sqrt{(1 - \beta^2)^2 + 4\zeta^2\beta^2}} \quad (1)$$

For $\beta = 1$ one has $|X| = 0.58$ cm, i.e.

$$\frac{0.58k}{|F|} \text{ cm} = \frac{1}{2\zeta} \quad (2)$$

For $\beta = 0.8$ one has $|X| = 0.46$ cm, i.e.

$$\frac{0.46k}{|F|} \text{ cm} = \frac{1}{\sqrt{(1 - 0.8^2)^2 + 4\zeta^2 0.8^2}} = \frac{1}{\sqrt{0.1296 + 2.56\zeta^2}} \quad (3)$$

From (2) and (3) it follows that

$$\frac{0.58}{0.46} = \frac{\sqrt{0.1296 + 2.56\zeta^2}}{2\zeta} \quad (4)$$

which has the unique solution

$$\zeta = 0.1847 \quad (5)$$

PROBLEM 3

Even though the excitation $f(t)$ is periodic, it is too irregular to meet the requirements of the Fourier series expansion. Hence, the method resulting in the solution (2-95) cannot be used.

The time axis is positioned, so that $t = 0$ is placed at an impulse. Between the impulses the system is performing undamped eigenvibrations with the circular frequency

$$\omega_0 = \sqrt{\frac{k}{m}} \quad (1)$$

The motion in the interval $]0, T[$ is given as, see (2-9)

$$\left. \begin{aligned} x(t) &= A \cos(\omega_0 t - \Psi) \\ \dot{x}(t) &= -A\omega_0 \sin(\omega_0 t - \Psi) \end{aligned} \right\} \quad (2)$$

The unknown amplitude A and phase Ψ are determined, so that the motion becomes period, and the discontinuity requirements of the velocity at the impulses are met. At $t = 0^+$ (2) is given as

$$\left. \begin{aligned} x(0^+) &= A \cos \Psi \\ \dot{x}(0^+) &= A\omega_0 \sin \Psi \end{aligned} \right\} \quad (3)$$

The next impulse arrives at the time $t = T$. At the time $t = T^-$ (2) provides

$$\left. \begin{aligned} x(T^-) &= A \cos(\omega_0 T - \Psi) \\ \dot{x}(T^-) &= -A\omega_0 \sin(\omega_0 T - \Psi) \end{aligned} \right\} \quad (4)$$

The impulse at the time $t = T$ does not affect the displacement, whereas the velocity has increased with $\frac{I}{m}$, cf. (2-103). Consequently, one has at the time $t = T^+$

$$\left. \begin{aligned} x(T^+) &= x(T^-) = A \cos(\omega_0 T - \Psi) \\ \dot{x}(T^+) &= \dot{x}(T^-) + \frac{I}{m} = -A\omega_0 \sin(\omega_0 T - \Psi) + \frac{I}{m} \end{aligned} \right\} \quad (5)$$

If the stationary motion is periodic with the period T , then $x(0^+) = x(T^+)$ and $\dot{x}(0^+) = \dot{x}(T^+)$. From (3) and (5) it then follows that

$$\left. \begin{aligned} A \cos \Psi &= A \cos(\omega_0 T - \Psi) \\ A\omega_0 \sin \Psi &= -A\omega_0 \sin(\omega_0 T - \Psi) + \frac{I}{m} \end{aligned} \right\} \Rightarrow$$

$$\left. \begin{aligned} \cos(\omega_0 T - \Psi) - \cos \Psi &= 0 \\ \sin(\omega_0 T - \Psi) + \sin \Psi &= \frac{I}{mA\omega_0} \end{aligned} \right\} \Rightarrow$$

$$\left. \begin{aligned} \sin\left(\frac{\omega_0 T}{2}\right) \sin\left(\frac{\omega_0 T}{2} - \Psi\right) &= 0 \\ \sin\left(\frac{\omega_0 T}{2}\right) \cos\left(\frac{\omega_0 T}{2} - \Psi\right) &= \frac{I}{2mA\omega_0} \end{aligned} \right\} \quad (6)$$

The 1st equation of (6) implies $\sin\left(\frac{\omega_0 T}{2}\right) = 0 \vee \sin\left(\frac{\omega_0 T}{2} - \Psi\right) = 0$. $\sin\left(\frac{\omega_0 T}{2}\right) = 0$ implies that the 2nd equation cannot be fulfilled. Consequently, the solution is

$$\begin{aligned} \sin\left(\frac{\omega_0 T}{2} - \Psi\right) &= 0 \Rightarrow \\ \Psi &= \frac{\omega_0 T}{2} + n\pi, \quad n = 0, \pm 1, \pm 2, \dots \end{aligned} \quad (7)$$

The solution for the amplitude can then be written

$$A = \frac{I}{2m\omega_0 \sin\left(\frac{\omega_0 T}{2}\right) \cos(n\pi)} \quad (8)$$

It follows from (8) that $A = \infty$, if $\frac{\omega_0 T}{2} = m\pi$, $m = 1, 2, \dots$. Introducing $\omega = \frac{2\pi}{T}$, infinite amplitudes are then attained at the frequency ratios

$$\frac{\omega_0}{\omega} = m, \quad m = 1, 2, \dots \quad (9)$$

By inserting (7) and (8) into (2) the following solution in the interval $]0, T[$ is obtained

$$\left. \begin{aligned} \frac{m\omega_0}{I}x(t) &= \frac{\cos\left(\omega_0 t - \frac{\omega_0 T}{2} - n\pi\right)}{2 \sin\left(\frac{\omega_0 T}{2}\right) \cos(n\pi)} = \frac{\cos\left(\omega_0 t - \frac{\omega_0 T}{2}\right)}{2 \sin\left(\frac{\omega_0 T}{2}\right)} \\ \frac{m}{I}\dot{x}(t) &= -\frac{\sin\left(\omega_0 t - \frac{\omega_0 T}{2} - n\pi\right)}{2 \sin\left(\frac{\omega_0 T}{2}\right) \cos(n\pi)} = -\frac{\sin\left(\omega_0 t - \frac{\omega_0 T}{2}\right)}{2 \sin\left(\frac{\omega_0 T}{2}\right)} \end{aligned} \right\} \quad (10)$$

The last statements of (10) follow from the trigonometrical identities $\cos(x - n\pi) = \cos(x)\cos(n\pi) + \sin(x)\sin(n\pi) = \cos(x)\cos(n\pi)$ and $\sin(x - n\pi) = \sin(x)\cos(n\pi) - \cos(x)\sin(n\pi) = \sin(x)\cos(n\pi)$. For other intervals than $]0, T[$ the solution is constructed from (10) using the periodicity. Below, the non-dimensional responses (10) have been plotted against the non-dimensional time $\frac{t}{T}$ for the cases $\omega_0 T = \frac{\pi}{2}$ and $\omega_0 T = \frac{5\pi}{2}$.

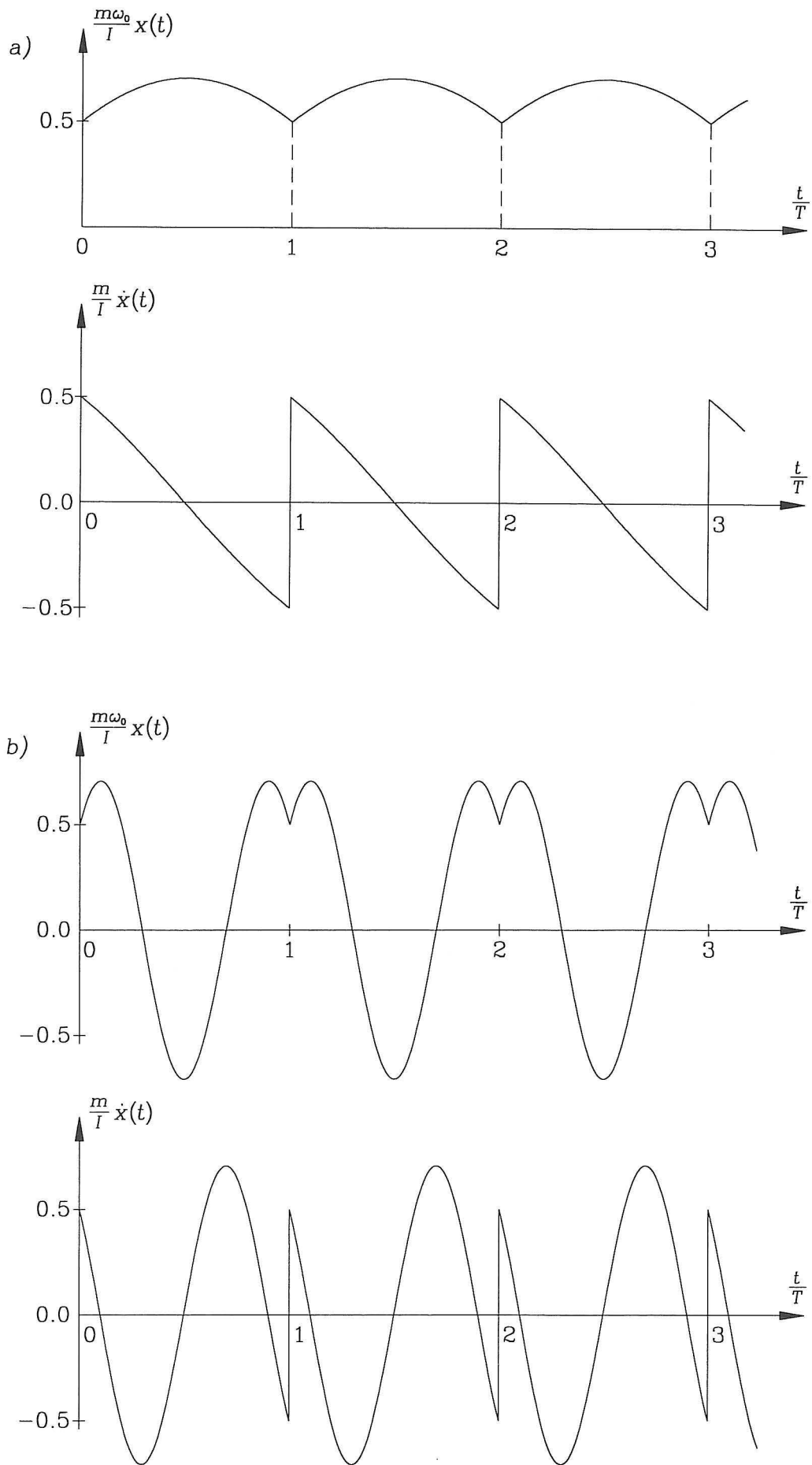


Fig. 1: Displacement and velocity responses. a) $\omega_0 T = \frac{\pi}{2}$. b) $\omega_0 T = \frac{5\pi}{2}$.

PROBLEM 4

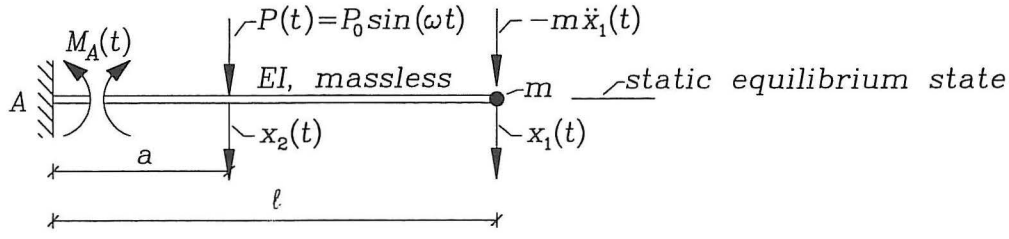


Fig. 1: Forces on the beam.

Since the beam is massless the system has but a single degree of freedom, which is selected as the vertical displacement $x_1(t)$ of the mass from the static equilibrium state. The vertical displacement at the force is termed $x_2(t)$. According to d'Alembert's principle an inertial load $-m\ddot{x}_1$ is applied to the mass, and the equation of motion can be derived from static calculations on the system. The displacement $x_1(t)$ is made up of displacement contributions from $P(t)$ and the inertial load, i.e.

$$x_1(t) = \delta_{11}(-m\ddot{x}_1) + \delta_{12}P(t) \Rightarrow$$

$$m\ddot{x}_1 + \frac{1}{\delta_{11}}x_1(t) = \frac{\delta_{12}}{\delta_{11}}P_0 \sin(\omega t) \quad , \quad t > 0 \quad (1)$$

Since the system starts from the rest the initial conditions of (1) read

$$x_1(0) = 0 \quad , \quad \dot{x}_1(0) = 0 \quad (2)$$

The flexibility coefficients δ_{11} and δ_{12} become, see (B-5)

$$\left. \begin{aligned} \delta_{11} &= \frac{l^3}{3EI} \\ \delta_{12} &= \frac{a^2(3l - a)}{6EI} \end{aligned} \right\} \quad (3)$$

Inserting (3) into (1) provides the differential equation for the motion

$$\left. \begin{aligned} \ddot{x}_1 + \omega_0^2 x_1 &= f_0 \sin(\omega t) \quad , \quad t > 0 \\ x_1(0) &= 0 \quad , \quad \dot{x}_1(0) = 0 \end{aligned} \right\} \quad (4)$$

$$\omega_0 = \sqrt{\frac{1}{\delta_{11}m}} = \sqrt{\frac{3EI}{ml^3}} \quad (5)$$

$$f_0 = \frac{\delta_{12}}{\delta_{11}} \frac{P_0}{m} = \frac{a^2(3l - a)}{2l^3} \frac{P_0}{m} \quad (6)$$

The solution of (4) reads

$$x_1(t) = f_0 \frac{\left(\sin(\omega t) - \frac{\omega}{\omega_0} \sin(\omega_0 t) \right)}{\omega_0^2 - \omega^2} \quad (7)$$

The validity of (7) is proved by inserting into (4). The displacement response is requested for the resonance case $\omega = \omega_0$. As seen from (7), the numerator as well as the denominator becomes singular in this case. Consequently, a limit value analysis of (7) as $\omega \rightarrow \omega_0$ must be performed. $\omega = \omega_0 + \Delta\omega$ is introduced, and the following 1st order Taylor-expansion in $\Delta\omega$ is performed

$$\sin((\omega_0 + \Delta\omega)t) = \sin(\omega_0 t) + t \cos(\omega_0 t) \Delta\omega + O(\Delta\omega^2) \quad (8)$$

where the order notation $O(x)$ means that $O(x) \leq A|x|$, A being a positive constant. (7) can then be written

$$x_1(t) = f_0 \frac{\sin(\omega_0 t) + t \cos(\omega_0 t) \Delta\omega - \left(1 + \frac{\Delta\omega}{\omega_0}\right) \sin(\omega_0 t) + O(\Delta\omega^2)}{\omega_0^2 - \omega^2 - 2\omega_0 \Delta\omega - \Delta\omega^2} \quad (9)$$

which provides the following limit as $\Delta\omega \rightarrow 0$

$$x_1(t) = -f_0 \frac{\omega_0 t \cos(\omega_0 t) - \sin(\omega_0 t)}{2\omega_0^2} \quad (10)$$

The moment at point A becomes, see fig. 1

$$M_A(t) = -aP_0 \sin(\omega_0 t) + m\ddot{x}_1(t) \cdot l \quad (11)$$

From (1) and (10) it follows that

$$\begin{aligned} m\ddot{x}_1 &= -\frac{1}{\delta_{11}} x_1(t) + \frac{\delta_{12}}{\delta_{11}} P_0 \sin(\omega_0 t) = \\ &= \frac{1}{\delta_{11}} f_0 \frac{\omega_0 t \cos(\omega_0 t) - \sin(\omega_0 t)}{2\omega_0^2} + \frac{\delta_{12}}{\delta_{11}} P_0 \sin(\omega_0 t) = \\ &= \frac{1}{2} \left(\omega_0 t \cos(\omega_0 t) + \sin(\omega_0 t) \right) \frac{\delta_{12}}{\delta_{11}} P_0 \end{aligned} \quad (12)$$

where it has been used that $\frac{f_0}{\delta_{11}\omega_0^2} = \frac{\delta_{12}}{\delta_{11}} \frac{P_0}{\delta_{11}m\omega_0^2} = \frac{\delta_{12}}{\delta_{11}} P_0$, see eqs. (5) and (6). Inserting (12) into (11) finally provides

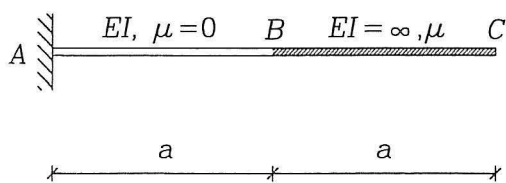
$$M_A(t) = -P_0 a \left(\sin(\omega_0 t) - \left(\sin(\omega_0 t) + \omega_0 t \cos(\omega_0 t) \right) \frac{(3l - a)a}{4l^2} \right) \quad (13)$$

1.4 Lecture 4

SUPPLEMENTARY PROBLEMS

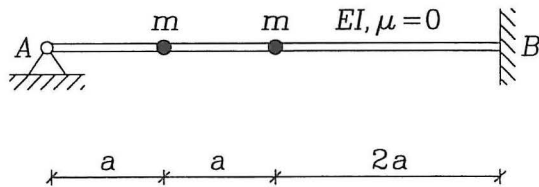
- May 14, 1987. Problem 1, question 1.
- June 26, 1987. Problem 1, questions 1 and 2.
- June 20, 1988. Problem 1, question 2.
- June 20, 1988. Problem 3, question 1.
- September 14, 1988. Problem 1, question 2.
- June 16, 1989. Problem 1, question 1.
- September 5, 1990. Problem 2, question 1.
- October 3, 1990. Problem 2.
- June 21, 1991. Problem 2, question 1.
- September 4, 1991. Problem 1, question 1.
- June 12, 1992. Problem 2.
- June 16, 1993. Problem 2, question 1.
- June 16, 1993. Problem 3, question 1.
- August 31, 1993. Problem 2.
- August 31, 1993. Problem 3.
- August 24, 1994. Problem 2.
- June 21, 1996. Problem 2, question 1.
- August 26, 1996. Problem 1, question 1.
- August 26, 1996. Problem 2, questions 1 and 2.
- June 22, 1998. Problem 1.

PROBLEM 1



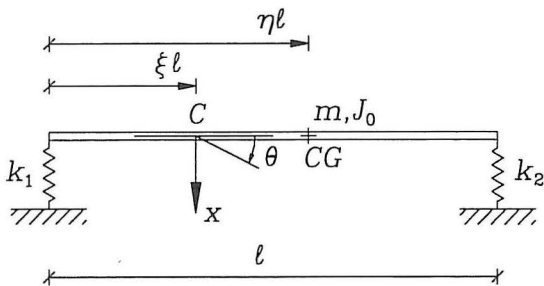
The beam in the figure shows a horizontal rectilinear beam ABC . The sub-beam AB is a massless, homogeneous Bernoulli-Euler beam of bending stiffness EI . The beam BC is infinitely stiff with constant mass per unit length μ . The length of both sub-beams is a . The beam is fixed at point A and is free at point C . Determine the undamped circular eigenfrequencies and eigenmodes of the structure. Only small vertical vibrations from the static equilibrium state are considered.

PROBLEM 2



The figure shows a homogeneous, massless Bernoulli-Euler beam AB . The length of the beam is $4a$ and the bending stiffness is EI . The beam is fixed, simply supported at point A and fixed at point B . Point masses of magnitude m are placed at the middle and at the quarter point of the beam. Determine the undamped circular eigenfrequencies and the eigenmodes of the system.

PROBLEM 3



The figure shows an infinitely stiff beam of the length l , supported by linear elastic springs with the spring constants k_1 and k_2 . The mass of the beam is m and the mass moment of inertia around the centre of gravity CG is J_0 . CG is placed at a distance ηl from the left end section where $\eta \in]0, 1[$. C denotes an arbitrary point on the beam, placed at the distance ξl from the left end section, where $\xi \in]0, 1[$. The system has 2 degrees of freedom, which are selected as the vertical displacement x and the rotation θ in the counter-clockwise direction of the point C .

Question 1:

Determine the equations of motion for the system.

Question 2:

Determine ξ in such a way that the mass matrix or the stiffness matrix becomes diagonal.

SOLUTIONS

PROBLEM 1

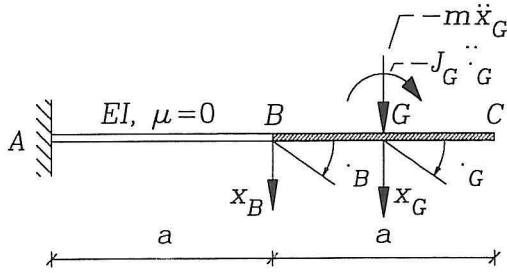


Fig. 1: Inertial loads on beam and definition of degrees of freedom.

Since the beam AB is massless, and the beam BC is infinitely stiff, the system has 2 degrees of freedom, which are selected as the vertical displacement x_B and the rotation θ_B of point B with signs as defined in fig. 1.

The mass centre of gravity G of beam BC then attains the displacement x_G and the rotation θ_G defined as

$$x_G = x_B + \theta_B \frac{a}{2} \quad (1)$$

$$\theta_G = \theta_B \quad (2)$$

The inertial load of beam BC is statically equivalent to a force $-m\ddot{x}_G$ and a moment $-J_G\ddot{\theta}_G$ applied at the centre of gravity and with signs as shown in fig. 1. $m = \mu a$ is the mass of the beam BC and $J_G = \frac{1}{12}\mu a^3$ is the mass moment of inertia referred to point G . These loads form the only excitation of the system during eigenvibrations. Consequently, the displacements x_B and θ_B are caused by these loads. Hence

$$\left. \begin{aligned} x_B &= \delta_{x_B x_G} \left(-\mu a \left(\ddot{x}_B + \ddot{\theta}_B \frac{a}{2} \right) \right) + \delta_{x_B \theta_G} \left(-\frac{1}{12} \mu a^3 \ddot{\theta}_B \right) \\ \theta_B &= \delta_{\theta_B x_G} \left(-\mu a \left(\ddot{x}_B + \ddot{\theta}_B \frac{a}{2} \right) \right) + \delta_{\theta_B \theta_G} \left(-\frac{1}{12} \mu a^3 \ddot{\theta}_B \right) \end{aligned} \right\} \quad (3)$$

where the coefficient of influence $\delta_{x_B x_G}$, $\delta_{x_B \theta_G}$, $\delta_{\theta_B x_G}$ and $\delta_{\theta_B \theta_G}$ specify the displacement or rotation at point B from a unit force or unit moment at point G . From

conventional static methods it follows that

$$\left. \begin{aligned} \delta_{x_B x_G} &= \frac{7}{12} \frac{a^3}{EI} \\ \delta_{\theta_B x_G} &= \frac{a^2}{EI} \\ \delta_{x_B \theta_G} &= \frac{a^2}{2EI} \\ \delta_{\theta_B \theta_G} &= \frac{a}{EI} \end{aligned} \right\} \quad (4)$$

Then, (3) can be written as follows

$$\left. \begin{aligned} x_B &= \frac{\mu a^4}{EI} \left(-\frac{7}{12} \ddot{x}_B - \frac{1}{3} a \ddot{\theta}_B \right) \\ a \theta_B &= \frac{\mu a^4}{EI} \left(-\ddot{x}_B - \frac{7}{12} a \ddot{\theta}_B \right) \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} \ddot{x}_B \\ a \ddot{\theta}_B \end{bmatrix} + 12 \frac{EI}{\mu a^4} \begin{bmatrix} 7 & -4 \\ -12 & 7 \end{bmatrix} \begin{bmatrix} x_B \\ a \theta_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (5)$$

The eigenvibrations are given as

$$\begin{bmatrix} x_B(t) \\ a \theta_B(t) \end{bmatrix} = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \cos(\omega t) \quad (6)$$

where the eigenmode is the solution to the linear homogeneous equations

$$\begin{bmatrix} 7 - \lambda & -4 \\ -12 & 7 - \lambda \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (7)$$

$$\lambda = \frac{1}{12} \frac{\mu a^4 \omega^2}{EI} \quad (8)$$

The characteristic equation becomes

$$\det \left(\begin{bmatrix} 7 - \lambda & -4 \\ -12 & 7 - \lambda \end{bmatrix} \right) = \lambda^2 - 14\lambda + 1 = 0 \Rightarrow$$

$$\left. \begin{aligned} \lambda_1 \\ \lambda_2 \end{aligned} \right\} = 7 \mp \sqrt{48} \Rightarrow$$

$$\left. \begin{matrix} \omega_1 \\ \omega_2 \end{matrix} \right\} = \sqrt{84 \mp 12\sqrt{48}} \sqrt{\frac{EI}{\mu a^4}} \quad (9)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix} \quad , \quad j = 1, 2 \quad (10)$$

The first component $\Phi_1^{(j)}$ is determined from the first equation of (7)

$$(7 - \lambda_j)\Phi_1^{(j)} - 4 \cdot 1 = 0 \Rightarrow$$

$$\Phi_1^{(j)} = \frac{4}{7 - \lambda_j} = \begin{cases} \frac{\sqrt{3}}{3} & , \quad j = 1 \\ -\frac{\sqrt{3}}{3} & , \quad j = 2 \end{cases} \quad (11)$$

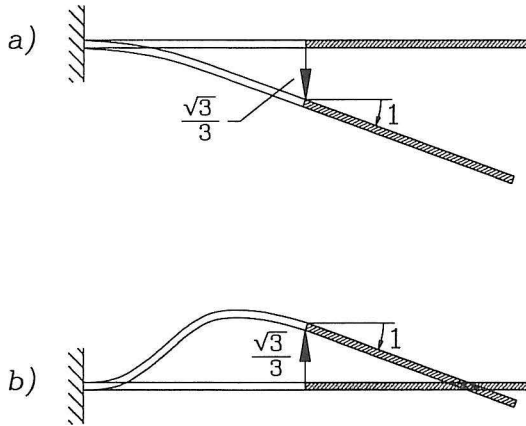


Fig. 2: a) 1st eigenmode. b) 2nd eigenmode.

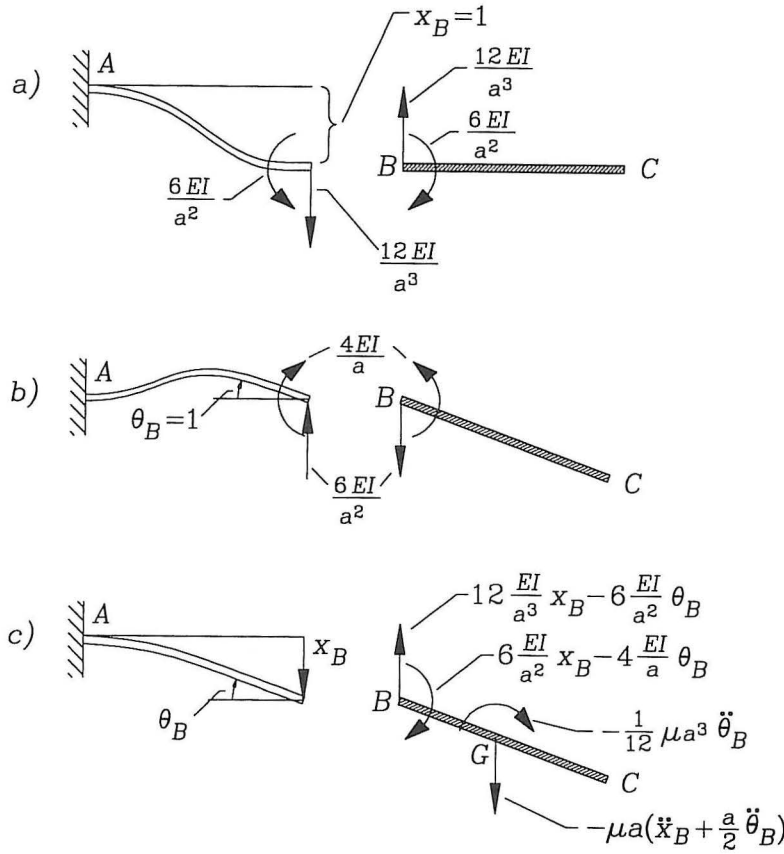


Fig. 3: a) Shear force and moment at point B from unit displacement $x_B = 1$. b) Shear force and moment at point B from unit rotation $\theta_B = 1$. c) Forces and moments at the free beam BC .

Alternatively, (5) can be derived, expressing that the shear force $12 \frac{EI}{a^3} x_B - 6 \frac{EI}{a^2} \theta_B$ and the bending moment $6 \frac{EI}{a^2} x_B - 4 \frac{EI}{a} \theta_B$, acting at point B on the free beam BC with signs as defined in fig. 3, must be in static equilibrium with the inertial load on the beam BC . The force and the moment equilibrium equations then read

$$\left. \begin{aligned} 12 \frac{EI}{a^3} x_B - 6 \frac{EI}{a^2} \theta_B + \mu a \left(\ddot{x}_B + \frac{a}{2} \ddot{\theta}_B \right) &= 0 \\ 6 \frac{EI}{a^2} x_B - 4 \frac{EI}{a} \theta_B - \frac{1}{12} \mu a^3 \ddot{\theta}_B - \mu a \left(\ddot{x}_B + \frac{a}{2} \ddot{\theta}_B \right) \frac{a}{2} &= 0 \end{aligned} \right\} \quad (12)$$

(12) is reduced to (5), if (12) is solved with respect to \ddot{x}_B and $a \ddot{\theta}_B$.

PROBLEM 2

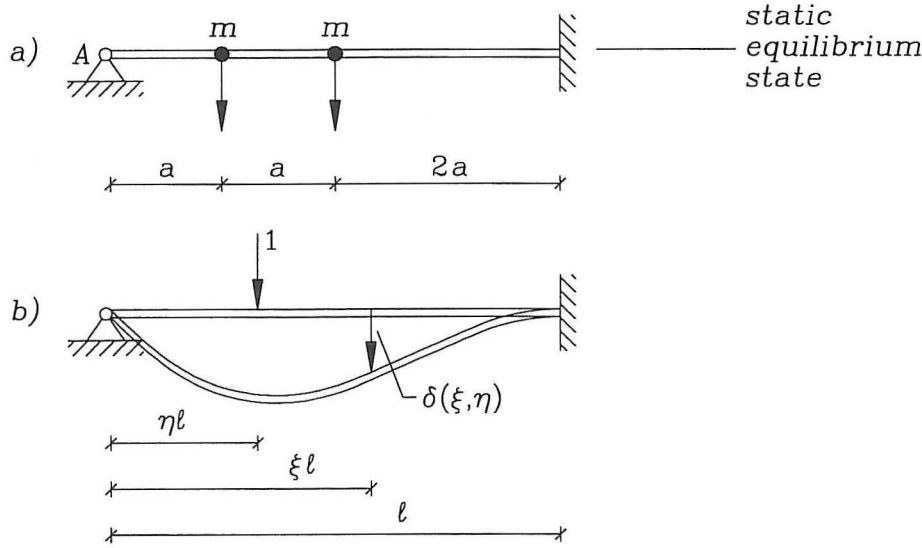


Fig. 1: a) 2 degrees-of-freedom system. b) Influence function for displacement at point ξl from unit force at ηl .

The beam is massless. Hence, the system has 2 degrees of freedom, which are selected as the vertical displacements $x_1(t)$ and $x_2(t)$ of the masses from the static equilibrium state with signs as defined in fig. 1a.

The system is statically indeterminate, and the determination of the flexibility coefficients δ_{ij} needs to become a little involved. Using standard static analysis techniques the coefficient of influence for the displacement at point ξl from a unit force at point $\eta l \leq \xi l$ is found as follows, see fig. 1b

$$\delta(\xi, \eta) = \frac{l^3}{12EI} \left((1 - \eta)^2 ((2 + \eta)(3\xi - \xi^3) - 6\eta) + 2(\xi - \eta)((\xi - \eta)^2 - 3(1 - \eta)^2) \right), \quad 0 \leq \eta \leq \xi \leq 1 \quad (1)$$

For the present case with $l = 4a$, $\xi, \eta = 0.25, 0.50$ is found

$$\left. \begin{aligned} \delta_{11} &= \frac{351}{768} \frac{a^3}{EI} \\ \delta_{12} &= \frac{344}{768} \frac{a^3}{EI} \\ \delta_{22} &= \frac{448}{768} \frac{a^3}{EI} \end{aligned} \right\} \quad (2)$$

The mass and flexibility matrices of the system become, cf. (3-5), (3-6)

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad \mathbf{D} = \frac{a^3}{768EI} \begin{bmatrix} 351 & 344 \\ 344 & 448 \end{bmatrix} \quad (3)$$

The circular eigenfrequencies ω_j and the eigenmodes $\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix}$ then become, cf. (3-55)

$$\left(\mathbf{DM} - \frac{1}{\omega_j^2} \mathbf{I} \right) \Phi^{(j)} = \mathbf{0} \Rightarrow$$

$$\begin{bmatrix} 351\lambda_j - 1 & 344\lambda_j \\ 344\lambda_j & 448\lambda_j - 1 \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad j = 1, 2 \quad (4)$$

$$\lambda_j = \frac{ma^3\omega_j^2}{768EI}, \quad j = 1, 2 \quad (5)$$

The frequency condition becomes

$$(351\lambda_j - 1)(448\lambda_j - 1) - 344^2\lambda_j^2 = 0 \Rightarrow$$

$$\lambda_j = \begin{cases} \frac{799 - \sqrt{482753}}{77824}, & j = 1 \\ \frac{799 + \sqrt{482753}}{77824}, & j = 2 \end{cases} \quad (6)$$

$$\omega_j = \begin{cases} \sqrt{\frac{2397 - \sqrt{4344777}}{304}} \sqrt{\frac{EI}{ma^3}} \simeq 1.014 \sqrt{\frac{EI}{ma^3}}, & j = 1 \\ \sqrt{\frac{2397 + \sqrt{4344777}}{304}} \sqrt{\frac{EI}{ma^3}} \simeq 3.839 \sqrt{\frac{EI}{ma^3}}, & j = 2 \end{cases} \quad (7)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix}, \quad j = 1, 2 \quad (8)$$

The first component $\Phi_1^{(j)}$ is determined from the first equation of (4)

$$(351\lambda_j - 1)\Phi_1^{(j)} + 344\lambda_j \cdot 1 = 0 \Rightarrow$$

$$\Phi_1 = \frac{344\lambda_j}{1 - 351\lambda_j} = \begin{cases} 0.8689, & j = 1 \\ -1.1509, & j = 2 \end{cases} \quad (9)$$

The eigenmodes have been sketched below in fig. 2.

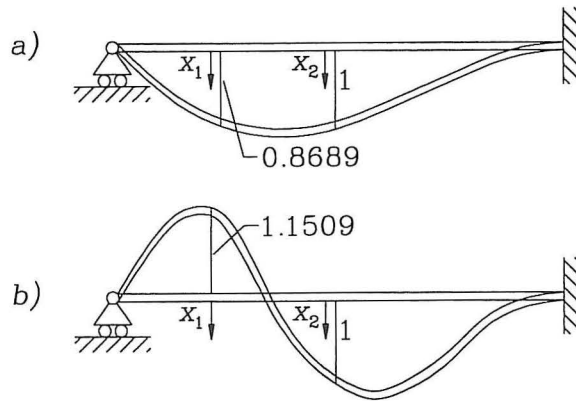


Fig. 2: a) 1st eigenmode. b) 2nd eigenmode.

PROBLEM 3

Question 1:

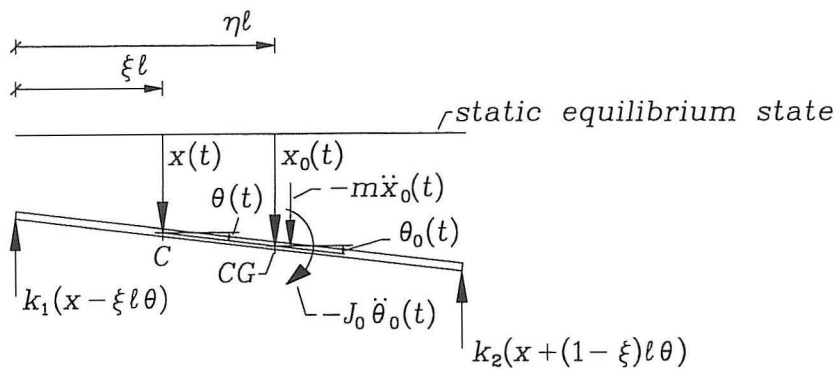


Fig. 1: Forces on the free beam.

The beam is infinitely stiff and then has but 2 degrees of freedom, which are selected as the vertical displacement $x(t)$ and the rotation $\theta(t)$ of point C from the static equilibrium state with signs as defined in fig. 1.

The vertical displacement $x_0(t)$ and the rotation $\theta_0(t)$ of the centre of gravity CG from the static equilibrium state with signs as defined in fig. 1 are introduced as auxiliary degrees of freedom. These are given as

$$x_0(t) = x(t) + (\eta - \xi)l\theta(t) \quad (1)$$

$$\theta_0(t) = \theta(t) \quad (2)$$

The beam is cut free from the springs, and the spring forces are applied as external forces with signs as shown in fig. 1. Further, the inertial force $-m\ddot{x}_0(t)$ and the inertial

moment $-J_0\ddot{\theta}_0(t)$ are applied as external loads acting in the centre of gravity according to d'Alembert's principle. The system must be in static equilibrium under the influence of these forces. Expressing the moment equation relative to point C , and using (1) and (2), the equations of motion become

$$\left. \begin{aligned} m(\ddot{x} + (\eta - \xi)l\ddot{\theta}) + k_1(x - \xi l\theta) + k_2(x + (1 - \xi)l\theta) &= 0 \\ J_0\ddot{\theta} + m(\ddot{x} + (\eta - \xi)l\ddot{\theta})(\eta - \xi)l - k_1(x - \xi l\theta)\xi l + k_2(x + (1 - \xi)l\theta)(1 - \xi)l &= 0 \end{aligned} \right\} \Rightarrow$$

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \quad (3)$$

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & (\eta - \xi)lm \\ (\eta - \xi)lm & J_0 + (\eta - \xi)^2 l^2 m \end{bmatrix} \quad (3a)$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -(\xi k_1 - (1 - \xi)k_2)l \\ -(\xi k_1 - (1 - \xi)k_2)l & (\xi^2 k_1 + (1 - \xi)^2 k_2)l^2 \end{bmatrix} \quad (3b)$$

Question 2:

From (3a) it follows that the mass matrix becomes diagonal if

$$\xi = \eta \quad (4)$$

From (3b) it follows that the stiffness matrix becomes diagonal if

$$\xi k_1 = (1 - \xi)k_2 \Rightarrow$$

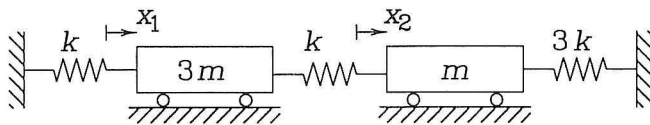
$$\xi = \frac{k_2}{k_1 + k_2} \quad (5)$$

(4) implies that C is placed in the centre of gravity. The point determined from (5) is denoted the elastic centre of gravity. A static force acting at this point will deform the beam without rotations.

SUPPLEMENTARY PROBLEMS

- June 20, 1988. Problem 3, question 2.
- September 14, 1988. Problem 1, question 1.
- June 16, 1993. Problem 3, question 2.
- June 22, 1994. Problem 2.
- August 24, 1994. Problem 3.
- June 19, 1997. Problem 1.

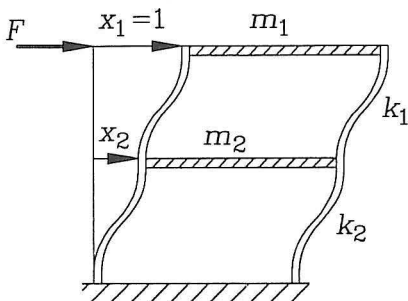
PROBLEM 1



Determine the undamped circular eigenfrequencies and eigenmodes of the 2 degrees-of-freedom system shown in the figure. Next, determine the undamped motion of the system in case of the initial conditions from the static equilibrium state

$$x_1(0) = x_2(0) = \dot{x}_2(0) = 0 \quad , \quad \dot{x}_1(0) = \dot{x}_{1,0}$$

PROBLEM 2



The 2-storey structure shown in the figure has massless columns and infinitely stiff storey beams. The shear stiffness of the columns in the upper and lower storeys is $k_1 = k$ and $k_2 = 2k$, respectively, and the corresponding storey masses are $m_1 = m$ and $m_2 = 2m$. The system is loaded statically at the upper storey by a horizontal force F , so that the displacement of this storey from the equilibrium state is $x_1 = 1$. At the time $t = 0$ the force F is momentarily removed. Determine the succeeding motion of the structure. Damping is ignored.

SOLUTIONS

PROBLEM 1

The mass and stiffness matrix of the system becomes, cf. (3-39)

$$\mathbf{M} = \begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & 4k \end{bmatrix} \quad (1)$$

The circular eigenfrequencies ω_j and the eigenmodes $\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix}$ are then determined from, cf. (3-61)

$$\begin{bmatrix} 2k - \omega_j^2 3m & -k \\ -k & 4k - \omega_j^2 m \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2)$$

The characteristic equation becomes

$$\begin{aligned} (2k - \omega_j^2 3m)(4k - \omega_j^2 m) - k^2 &= 0 \Rightarrow \\ \left. \begin{matrix} \omega_1^2 \\ \omega_2^2 \end{matrix} \right\} &= \frac{7 \mp \sqrt{28}}{3} \frac{k}{m} \\ \left. \begin{matrix} \omega_1 \\ \omega_2 \end{matrix} \right\} &= \sqrt{\frac{7 \mp \sqrt{28}}{3}} \omega_0, \quad \omega_0 = \sqrt{\frac{k}{m}} \end{aligned} \quad (3)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix}, \quad j = 1, 2 \quad (4)$$

The first component $\Phi_1^{(j)}$ is determined from the first equation of (2)

$$\begin{aligned} (2k - \omega_j^2 3m)\Phi_1^{(j)} - k \cdot 1 &= 0 \Rightarrow \\ \Phi_1^{(j)} = \frac{k}{2k - \omega_j^2 3m} &= \begin{cases} \frac{5+\sqrt{28}}{3} \simeq 3.431 & , \quad j = 1 \\ \frac{5-\sqrt{28}}{3} \simeq -0.097 & , \quad j = 2 \end{cases} \end{aligned} \quad (5)$$

The eigenvibration $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ due to the initial conditions $\mathbf{x}(0) = \mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\dot{\mathbf{x}}(0) = \dot{\mathbf{x}}_0 = \begin{bmatrix} \dot{x}_{1,0} \\ 0 \end{bmatrix}$ is given as, cf. (3-79)

$$\mathbf{x}(t) = a_1 \Phi^{(1)} \cos(\omega_1 t) + a_2 \Phi^{(2)} \cos(\omega_2 t) + b_1 \Phi^{(1)} \sin(\omega_1 t) + b_2 \Phi^{(2)} \sin(\omega_2 t) \quad (6)$$

The modal matrix becomes, cf. (3-83)

$$\mathbf{P} = \begin{bmatrix} \frac{5+\sqrt{28}}{3} & \frac{5-\sqrt{28}}{3} \\ 1 & 1 \end{bmatrix} \Rightarrow$$

$$\mathbf{P}^{-1} = \frac{3\sqrt{28}}{56} \begin{bmatrix} 1 & -\frac{5-\sqrt{28}}{3} \\ -1 & \frac{5+\sqrt{28}}{3} \end{bmatrix} \quad (7)$$

The expansion coefficients a_1, a_2, b_1, b_2 are given by (3-81), (3-82)

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{P}^{-1} \mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (8)$$

$$\begin{bmatrix} b_1 \omega_1 \\ b_2 \omega_2 \end{bmatrix} = \mathbf{P}^{-1} \dot{\mathbf{x}}_0 = \frac{3\sqrt{28}}{56} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \dot{x}_{1,0} \quad (9)$$

From (6), (8), (9) it follows that

$$\mathbf{x}(t) = \begin{bmatrix} \frac{5+\sqrt{28}}{3} \\ 1 \end{bmatrix} \frac{3\sqrt{28}}{56} \frac{\dot{x}_{1,0}}{\omega_1} \sin(\omega_1 t) - \begin{bmatrix} \frac{5-\sqrt{28}}{3} \\ 1 \end{bmatrix} \frac{3\sqrt{28}}{56} \frac{\dot{x}_{1,0}}{\omega_2} \sin(\omega_2 t) \simeq$$

$$\begin{bmatrix} 1.289 \\ 0.376 \end{bmatrix} \frac{\dot{x}_{1,0}}{\omega_0} \sin(0.755\omega_0 t) + \begin{bmatrix} 0.014 \\ -0.140 \end{bmatrix} \frac{\dot{x}_{1,0}}{\omega_0} \sin(2.024\omega_0 t) \quad , \quad t \geq 0 \quad (10)$$

PROBLEM 2

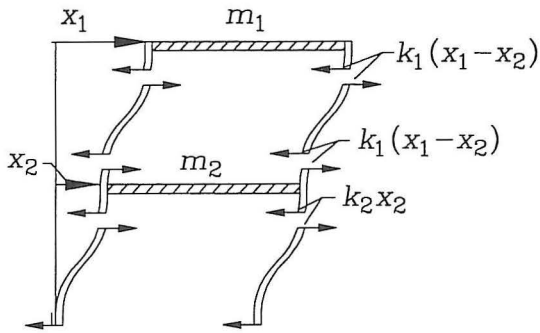


Fig. 1: Forces on free storey masses.

The storey masses are cut free, and the shear forces from the columns are applied as external forces with signs as shown in fig. 1. Newton's 2nd law for the free storey masses becomes

$$\left. \begin{aligned} m_1 \ddot{x}_1 &= -k_1(x_1 - x_2) \\ m_2 \ddot{x}_2 &= k_1(x_1 - x_2) - k_2 x_2 \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (1)$$

Inserting $k_1 = k$, $k_2 = 2k$ and $m_1 = m$, $m_2 = 2m$ the circular eigenfrequencies ω_j and the eigenmodes $\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix}$ then become, cf. (3-61)

$$\begin{bmatrix} k - \omega_j^2 m & -k \\ -k & 3k - \omega_j^2 2m \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2)$$

The frequency condition becomes

$$(k - \omega_j^2 m)(3k - \omega_j^2 2m) - k^2 = 0 \Rightarrow$$

$$\left. \begin{aligned} \omega_1^2 \\ \omega_2^2 \end{aligned} \right\} = \left\{ \begin{aligned} \frac{1}{2} \frac{k}{m} \\ 2 \frac{k}{m} \end{aligned} \right\} \Rightarrow$$

$$\left. \begin{aligned} \omega_1 \\ \omega_2 \end{aligned} \right\} = \left\{ \begin{aligned} \frac{\sqrt{2}}{2} \omega_0 \\ \sqrt{2} \omega_0 \end{aligned} \right\}, \quad \omega_0 = \sqrt{\frac{k}{m}} \quad (3)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix}, \quad j = 1, 2 \quad (4)$$

The first component $\Phi_1^{(j)}$ is determined from the first equation of (2)

$$(k - \omega_j^2 m) \Phi_1^{(j)} - k \cdot 1 = 0 \Rightarrow$$

$$\Phi_1^{(j)} = \frac{k}{k - \omega_j^2 m} = \begin{cases} 2 & , \quad j = 1 \\ -1 & , \quad j = 2 \end{cases} \quad (5)$$

The solution of (1) due to the initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(0) = \dot{\mathbf{x}}_0 = \mathbf{0}$ is requested. \mathbf{x}_0 is determined from the static equilibrium equation

$$\mathbf{K} \mathbf{x}_0 = \begin{bmatrix} F \\ 0 \end{bmatrix} \Rightarrow$$

$$\mathbf{x}_0 = \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} = \begin{bmatrix} k & -k \\ -k & 3k \end{bmatrix}^{-1} \begin{bmatrix} F \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} \frac{F}{k} \quad (6)$$

It is known that $x_{1,0} = 1 \Rightarrow \frac{F}{k} = \frac{2}{3}$. Hence

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} \quad (7)$$

The eigenvibrations follow from (3-79)

$$\mathbf{x}(t) = a_1 \Phi^{(1)} \cos(\omega_1 t) + a_2 \Phi^{(2)} \cos(\omega_2 t) + b_1 \Phi^{(1)} \sin(\omega_1 t) + b_2 \Phi^{(2)} \sin(\omega_2 t) \quad (8)$$

The modal matrix becomes, cf. (3-83)

$$\mathbf{P} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow$$

$$\mathbf{P}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \quad (9)$$

The expansion coefficients a_1, a_2, b_1, b_2 are given by (3-81), (3-82)

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{P}^{-1} \mathbf{x}_0 = \begin{bmatrix} \frac{4}{9} \\ -\frac{1}{9} \end{bmatrix} \quad (10)$$

$$\begin{bmatrix} \omega_1 b_1 \\ \omega_2 b_2 \end{bmatrix} = \mathbf{P}^{-1} \dot{\mathbf{x}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (11)$$

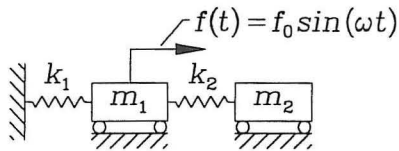
From (8), (10), (11) it follows that

$$\mathbf{x}(t) = \frac{4}{9} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cos(\omega_1 t) + \left(-\frac{1}{9}\right) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cos(\omega_2 t) =$$

$$\begin{bmatrix} \frac{8}{9} \\ \frac{4}{9} \end{bmatrix} \cos\left(\frac{\sqrt{2}}{2} \omega_0 t\right) + \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{9} \end{bmatrix} \cos(\sqrt{2} \omega_0 t) \quad , \quad t \geq 0 \quad (12)$$

1.6 Lecture 6

PROBLEM 1



The springs of the system shown in the figure are linear elastic with the indicated spring constants. The mass m_1 is subjected to a harmonic load $f(t) = f_0 \sin(\omega t)$, whereas the mass m_2 is unloaded. The system can only move in the direction of the springs. Determine the stationary motion of the system. Show that the mass m_1 is at rest, if k_2 and m_2 are selected, so that

$$\sqrt{\frac{k_2}{m_2}} = \omega$$

PROBLEM 2

June 26, 1987. Problem 1.

PROBLEM 3

Problem 1 in lecture 3 is considered once again. However, now the beam is a homogeneous massless Bernoulli-Euler beam with the bending stiffness EI . Answer the same question as in problem 1, lecture 3.

SOLUTIONS

PROBLEM 1

The solution of problem 1 has been included as example 3-11 in the textbook and will not be reiterated here.

PROBLEM 2

See solution at June 26, 1987. Problem 1.

PROBLEM 3

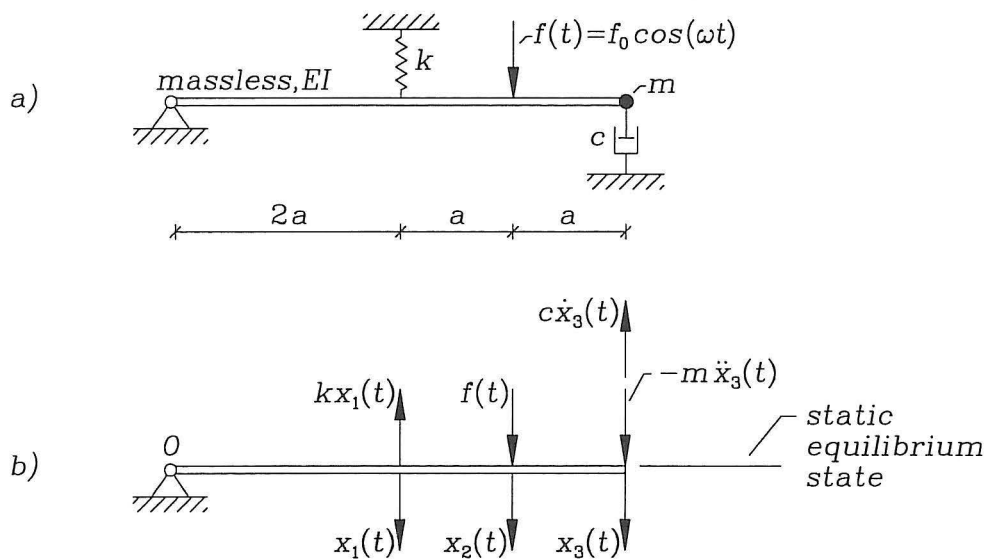


Fig. 1: a) Single degree-of-freedom system with massless, flexible beam. b) Forces on free beam.

The beam is still massless. Hence, the system still has but a single degree of freedom, which is selected as the vertical displacement $x_3(t)$ of the point mass m from the static equilibrium state. However, because of the flexible beam the problem is most easily solved applying the solution methods for a multiple degrees-of-freedom system. In this context, artificial degrees of freedom $x_1(t)$ and $x_2(t)$ are introduced, measuring the vertical displacement at the spring and the external force, as shown in fig. 1b. The beam is cut free from the spring and the damper, and the damper force $c\dot{x}_3(t)$ and the spring force $kx_1(t)$ are applied as external forces with signs as shown in fig. 1b. Further, the inertial load $-m\ddot{x}_3(t)$ is applied as an external load acting on the mass. Moment

equilibrium at point O then provides the equation of motion

$$(-m\ddot{x}_3 - c\dot{x}_3)4a + f(t) \cdot 3a - kx_1(t) \cdot 2a = 0 \quad (1)$$

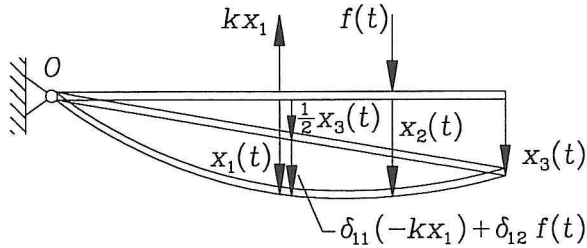


Fig. 2: Composition of displacement at the spring.

The displacement $x_1(t)$ at the spring is formed primarily by a stiff-body contribution of magnitude $\frac{1}{2}x_3(t)$, as considered in lecture 3, problem 1. On the top of this, elastic increments of magnitude $\delta_{11}(-kx_1)$ and $\delta_{12}f(t)$ from the spring force and the external force are present. δ_{11} and δ_{12} are the flexibility coefficients, when both ends of the beam are simply supported, as seen from the displacement curve shown in figure 2. It then follows that

$$x_1(t) = \frac{1}{2}x_3(t) + \delta_{11}(-kx_1(t)) + \delta_{12}f(t) \Rightarrow$$

$$x_1(t) = \frac{1}{1 + \delta_{11}k} \left(\frac{1}{2}x_3(t) + \delta_{12}f(t) \right) \quad (2)$$

Eliminating $x_1(t)$ in (1) by means of (2), the following equation of motion is obtained

$$\ddot{x}_3 + 2\zeta\omega_0\dot{x}_3 + \omega_0^2x_3 = \frac{F_0}{m} \cos(\omega t) \quad (3)$$

where

$$\omega_0^2 = \frac{k}{4(1 + \delta_{11}k)m} \Rightarrow$$

$$\omega_0 = \frac{1}{2} \sqrt{\frac{k}{(1 + \delta_{11}k)m}} \quad (4)$$

$$\zeta = \frac{c}{2m\omega_0} = c \sqrt{\frac{1 + \delta_{11}k}{mk}} \quad (5)$$

$$F_0 = \left(\frac{3}{4} - \frac{\delta_{12}k}{2(1 + \delta_{11}k)} \right) f_0 \quad (6)$$

The coefficients of influence become, see (B-1)

$$\left. \begin{aligned} \delta_{11} &= \frac{16}{12} \frac{a^3}{EI} \\ \delta_{12} &= \frac{11}{12} \frac{a^3}{EI} \end{aligned} \right\} \quad (7)$$

The decisive parameter reflecting the relative flexibility of the concentrated spring and the bending stiffness of the beam is $\delta_{11}k = \frac{16}{12} \frac{ka^3}{EI}$. The damping ratio of the system then depends on both k and the bending stiffness EI , as follows from (5).

The stationary motion of the mass m now becomes, cf. (2-64)

$$x_3(t) = |X_3| \cos(\omega t - \Psi_3) \quad (8)$$

$$|X_3| = \frac{F_0}{m \sqrt{(\omega_0^2 - \omega^2)^2 + 4\zeta^2 \omega_0^2 \omega^2}} \quad (9)$$

$$\tan(\Psi_3) = \frac{2\zeta \omega_0 \omega}{\omega_0^2 - \omega^2} \quad (10)$$

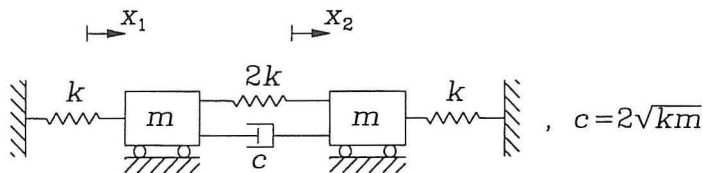
where ω_0 , ζ , F_0 are given by (4), (5), (6), (7).

1.7 Lecture 7

SUPPLEMENTARY PROBLEMS

- June 22, 1998. Problem 3, question 2.

PROBLEM 1

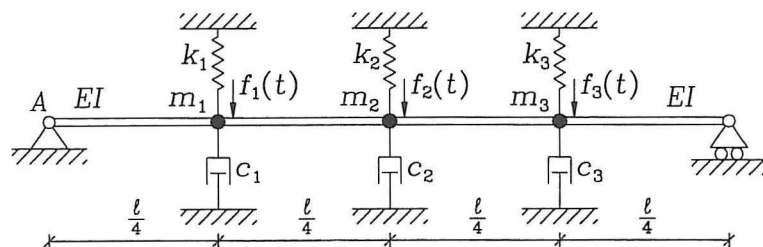


Determine the damped motion of the linear 2 degrees-of-freedom system shown in the figure in case of the initial conditions from the static equilibrium state

$$x_1(0) = x_{1,0} , \quad x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$$

Hint: Initially, check whether an expansion in the undamped eigenmodes can be applied.

PROBLEM 2



The figure shows a homogeneous massless Bernoulli-Euler beam of the length l and the bending stiffness EI . At the quarter points a linear elastic spring with a spring constant k_i , a linear viscous damper with a damper constant c_i , a point mass m_i and an external load $f_i(t)$ are present.

Question 1:

Determine the equations of motion of the system.

Question 2:

Determine the motion of the system in case $f_1(t) = f_3(t) \equiv 0$ and $f_2(t)$ is a static load of the magnitude f_0 , which is momentarily removed at $t = 0$.

The following parameter values are applied

$$k_1 = k_2 = k_3 = 2k_0$$

$$c_1 = c_2 = c_3 = c$$

$$m_1 = m_2 = m_3 = m$$

$$\text{where } k_0 = \frac{192}{7} \frac{EI}{l^3}, \quad c = 0.32 \sqrt{\frac{12}{7}} \sqrt{\frac{EI m}{l^3}}.$$

SOLUTIONS

PROBLEM 1

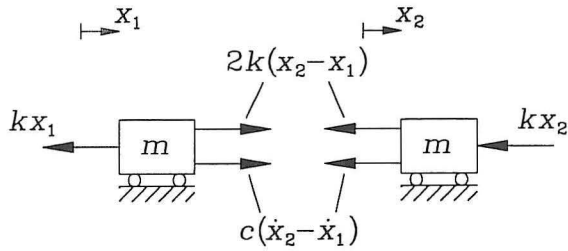


Fig. 1: Forces on free masses.

The masses are cut free, and the spring and the damping forces are applied as external forces as shown in fig. 1. Newton's 2nd law of motion provides

$$\left. \begin{aligned} m\ddot{x}_1 &= -kx_1 + 2k(x_2 - x_1) + c(\dot{x}_2 - \dot{x}_1) \\ m\ddot{x}_2 &= -kx_2 - 2k(x_2 - x_1) - c(\dot{x}_2 - \dot{x}_1) \end{aligned} \right\} \Rightarrow$$

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}, \quad t > 0 \quad (1)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 3k & -2k \\ -2k & 3k \end{bmatrix} \quad (2)$$

It follows from (2) that

$$\mathbf{C} = -\frac{c}{2m}\mathbf{M} + \frac{c}{2k}\mathbf{K} \quad (3)$$

Hence, the damping model is a Rayleigh damping model with $a_0 = -\frac{c}{2m}$ and $a_1 = \frac{c}{2k}$, cf. (3-284). Consequently, the decoupling condition (3-184) is fulfilled, and the expansion of the solution into undamped modal coordinates results in decoupled modal coordinate differential equations. Such an approach will then be followed.

The circular eigenfrequencies ω_j and the eigenmodes $\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix}$ are obtained as non-trivial solutions of the homogeneous linear equations, cf. (3-61)

$$\begin{bmatrix} 3k - \omega_j^2 m & -2k \\ -2k & 3k - \omega_j^2 m \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (4)$$

The characteristic equation becomes

$$\begin{aligned} (3k - \omega_j^2 m)(3k - \omega_j^2 m) - 4k^2 &= 0 \Rightarrow \\ \omega_j^2 &= \begin{cases} \frac{k}{m} & , \quad j = 1 \\ 5\frac{k}{m} & , \quad j = 2 \end{cases} \Rightarrow \\ \omega_j &= \begin{cases} \sqrt{\frac{k}{m}} & , \quad j = 1 \\ \sqrt{5}\sqrt{\frac{k}{m}} & , \quad j = 2 \end{cases} \end{aligned} \quad (5)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix} , \quad j = 1, 2 \quad (6)$$

The first component $\Phi_1^{(j)}$ is determined from the first equation of (4)

$$\begin{aligned} (3k - \omega_j^2 m)\Phi_1^{(j)} - 2k \cdot 1 &= 0 \Rightarrow \\ \Phi_1^{(j)} &= \frac{2k}{3k - \omega_j^2 m} = \begin{cases} 1 & , \quad j = 1 \\ -1 & , \quad j = 2 \end{cases} \end{aligned} \quad (7)$$

The modal masses become

$$M_j = \Phi^{(j)T} \mathbf{M} \Phi^{(j)} = 2m \quad , \quad j = 1, 2 \quad (8)$$

The displacement response is next expanded as a linear combination of the linear independent basis vectors $\Phi^{(1)}$ and $\Phi^{(2)}$

$$\mathbf{x}(t) = q_1(t)\Phi^{(1)} + q_2(t)\Phi^{(2)} \quad (9)$$

The differential equations for the modal coordinates are given by (3-181), (3-182), (3-183) with $F_i(t) \equiv 0$

$$\ddot{q}_i + 2\omega_i \left(\zeta_i \dot{q}_i + \sum_{\substack{j=1 \\ j \neq i}}^2 \sqrt{\frac{\omega_j M_j}{\omega_i M_i}} \zeta_{ij} \dot{q}_j \right) + \omega_i^2 q_i = 0, \quad i = 1, 2 \quad (10)$$

$$\zeta_j = \frac{\Phi^{(j)T} \mathbf{C} \Phi^{(j)}}{2\omega_j M_j} = \begin{cases} 0 & , \quad j = 1 \\ \frac{\sqrt{5}}{5} \frac{c}{\sqrt{km}} = 2\frac{\sqrt{5}}{5} & , \quad j = 2 \end{cases} \quad (11)$$

$$\zeta_{12} = \frac{\Phi^{(1)T} \mathbf{C} \Phi^{(2)}}{2\sqrt{\omega_1 \omega_2 M_1 M_2}} = 0 \quad (12)$$

In (11) the parameter value $c = 2\sqrt{km}$ has been used. (12) confirms the modal decoupling already discovered from the expansion (3). The initial values related to (10) follow from (3-175) with $\mathbf{x}_0^T = [x_{1,0}, 0]$, $\dot{\mathbf{x}}_0^T = [0, 0]$

$$q_i(0) = \frac{1}{M_i} \Phi^{(i)T} \mathbf{M} \mathbf{x}_0 = \begin{cases} \frac{1}{2} x_{1,0} & , \quad i = 1 \\ -\frac{1}{2} x_{1,0} & , \quad i = 2 \end{cases} \quad (13)$$

$$\dot{q}_i(0) = \frac{1}{M_i} \Phi^{(i)T} \mathbf{M} \dot{\mathbf{x}}_0 = 0, \quad i = 1, 2 \quad (14)$$

The solution of (10) then becomes, cf. (3-188), (3-189)

$$\left. \begin{aligned} q_1(t) &= q_1(0) \cos(\omega_1 t) + \frac{\dot{q}_1(0)}{\omega_1} \sin(\omega_1 t) \\ q_2(t) &= e^{-\zeta_2 \omega_2 t} \left(q_2(0) \cos(\omega_{d,2} t) + \frac{\dot{q}_2(0) + \zeta_2 \omega_2 q_2(0)}{\omega_{d,2}} \sin(\omega_{d,2} t) \right) \end{aligned} \right\} \quad (15)$$

where, cf. (5), (11)

$$\omega_{d,2} = \omega_2 \sqrt{1 - \zeta_2^2} = \sqrt{5} \sqrt{\frac{k}{m}} \sqrt{1 - \left(2\frac{\sqrt{5}}{5}\right)^2} = \sqrt{\frac{k}{m}} = \omega_1 \quad (16)$$

$$\zeta_2 \omega_2 = 2 \frac{\sqrt{5}}{5} \cdot \sqrt{5} \sqrt{\frac{k}{m}} = 2 \sqrt{\frac{k}{m}} = 2\omega_1 \quad (17)$$

Applying the initial conditions (13) and (14) in (15) and inserting the result into (9), the resulting damped eigenvibrations are obtained

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t) - \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2\omega_1 t} (\cos(\omega_1 t) + 2 \sin(\omega_1 t)) \right) \frac{x_{1,0}}{2} \quad (18)$$

Note:

During eigenvibrations in the first eigenmode, the displacement of the two masses is completely identical. Hence, the relative velocity $\dot{x}_2 - \dot{x}_1 = 0$, leaving the damper inactive. No dissipation of energy is performed during this eigenvibration, and $\zeta_1 = 0$. At the eigenvibrations in the second eigenmode the masses move opposite to each other. The damper then becomes active only if the eigenvibrations in this mode are activated.

PROBLEM 2

Question 1:

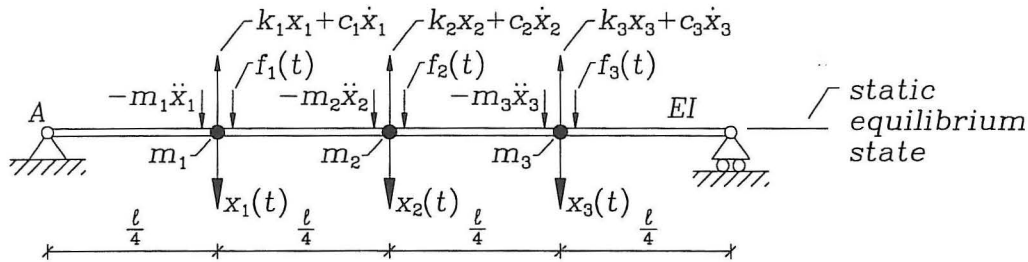


Fig. 1: Forces on the free beam.

The beam is massless. Hence, the system has 3 degrees of freedom which are selected as the vertical displacements $x_1(t)$, $x_2(t)$ and $x_3(t)$ of the masses from the static equilibrium state with signs as defined in fig. 1. The springs and the dampers are cut free from the beam, and the spring and damper forces are applied as external forces. Finally, the inertial forces $-m_i \ddot{x}_i$ are applied as external forces and with signs as shown in fig. 1. The equations of motion then become, cf. (3-1)

$$x_i = \sum_{j=1}^3 \delta_{ij} (f_j(t) - m_j \ddot{x}_j - c_j \dot{x}_j - k_j x_j) \quad (1)$$

The flexibility matrix \mathbf{D} with components given by the coefficient of influence for the displacement δ_{ij} for the free beam can be calculated as follows, cf. (B-1)

$$\mathbf{D} = \frac{l^3}{768EI} \begin{bmatrix} 9 & 11 & 7 \\ 11 & 16 & 11 \\ 7 & 11 & 9 \end{bmatrix} \quad (2)$$

The stiffness matrix from the beam stiffness alone then becomes

$$\mathbf{D}^{-1} = \frac{192}{7} \frac{EI}{l^3} \begin{bmatrix} 23 & -22 & 9 \\ -22 & 32 & -22 \\ 9 & -22 & 23 \end{bmatrix} \quad (3)$$

On matrix form, after premultiplication of \mathbf{D}^{-1} , (1) can be written as follows

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \quad , \quad t > 0 \quad (4)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}, \mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix} \quad (5)$$

$$\mathbf{K} = \mathbf{D}^{-1} + \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} = \begin{bmatrix} 23k_0 + k_1 & -22k_0 & 9k_0 \\ -22k_0 & 32k_0 + k_2 & -22k_0 \\ 9k_0 & -22k_0 & 23k_0 + k_3 \end{bmatrix} \quad (6)$$

$$k_0 = \frac{192}{7} \frac{EI}{l^3} \quad (7)$$

In the present case the following data are specified $k_1 = k_2 = k_3 = 2k_0$. Hence, the stiffness matrix becomes

$$\mathbf{K} = \begin{bmatrix} 25 & -22 & 9 \\ -22 & 34 & -22 \\ 9 & -22 & 25 \end{bmatrix} k_0 \quad (8)$$

Question 2:

$\mathbf{f}(t) \equiv 0$, $t > 0$. However, $f_2(t) = f_0$, $t < 0$, which causes the initial displacement \mathbf{x}_0 , determined from the static equilibrium equation as follows

$$\mathbf{K}\mathbf{x}_0 = \begin{bmatrix} 0 \\ f_0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\mathbf{x}_0 = \mathbf{K}^{-1} \begin{bmatrix} 0 \\ f_0 \\ 0 \end{bmatrix} = \frac{1}{1504k_0} \begin{bmatrix} 183 & 176 & 89 \\ 176 & 272 & 176 \\ 89 & 176 & 183 \end{bmatrix} \begin{bmatrix} 0 \\ f_0 \\ 0 \end{bmatrix} = \frac{7}{18048} \begin{bmatrix} 11 \\ 17 \\ 11 \end{bmatrix} \frac{f_0 l^3}{EI} \quad (9)$$

The initial velocity is given as

$$\dot{\mathbf{x}}_0 = \mathbf{0} \quad (10)$$

$c_1 = c_2 = c_3 = c$ and $m_1 = m_2 = m_3 = m$. Hence,

$$\mathbf{C} = \frac{c}{m} \mathbf{M} \quad (11)$$

(11) is a special case of Rayleigh-damping with $a_0 = \frac{c}{m}$ and $a_1 = 0$, cf. (3-284). Then, the decoupling condition (3-184) is fulfilled, and the solution of (4) with the initial conditions (9), (10) using expansion into undamped eigenmodes becomes possible. The circular eigenfrequencies and the eigenmodes are obtained as non-trivial solutions of the homogeneous linear equations, cf. (3-42), (5), (8)

$$(\mathbf{K} - \omega_j^2 \mathbf{M}) \Phi^{(j)} = \mathbf{0} \Rightarrow$$

$$\begin{bmatrix} 25 - \lambda_j & -22 & 9 \\ -22 & 34 - \lambda_j & -22 \\ 9 & -22 & 25 - \lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \\ \Phi_3^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (12)$$

$$\lambda_j = \frac{7}{192} \frac{\omega_j^2 m l^3}{EI} \quad (13)$$

The characteristic equation, expressing that the determinant of the coefficient matrix of (13) is equal to zero becomes

$$\lambda_j^3 - 84\lambda_j^2 + 1276\lambda_j - 3008 = 0 \Rightarrow$$

$$\lambda_j = \begin{cases} 34 - \sqrt{968} & , \quad j = 1 \\ 16 & , \quad j = 2 \\ 34 + \sqrt{968} & , \quad j = 3 \end{cases} \quad (14)$$

$$\omega_j = \begin{cases} \sqrt{\frac{192}{7} (34 - \sqrt{968})} \sqrt{\frac{EI}{ml^3}} = 8.899 \sqrt{\frac{EI}{ml^3}} & , \quad j = 1 \\ \sqrt{\frac{192}{7} 16} \sqrt{\frac{EI}{ml^3}} = 20.949 \sqrt{\frac{EI}{ml^3}} & , \quad j = 2 \\ \sqrt{\frac{192}{7} (34 + \sqrt{968})} \sqrt{\frac{EI}{ml^3}} = 42.260 \sqrt{\frac{EI}{ml^3}} & , \quad j = 3 \end{cases} \quad (15)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \\ 1 \end{bmatrix} \quad (16)$$

The first and second components are determined from the first 2 equations of (12)

$$\left. \begin{aligned} (25 - \lambda_j) \Phi_1^{(j)} - 22\Phi_2^{(j)} + 9 \cdot 1 &= 0 \\ -22\Phi_1^{(j)} + (34 - \lambda_j) \Phi_2^{(j)} - 22 \cdot 1 &= 0 \end{aligned} \right\} \Rightarrow$$

$$\Phi_1^{(j)} = \frac{178 + 9\lambda_j}{\lambda_j^2 - 59\lambda_j + 366} = \begin{cases} 1 & , \quad j = 1 \\ -1 & , \quad j = 2 \\ 1 & , \quad j = 3 \end{cases} \quad (17a)$$

$$\Phi_2^{(j)} = \frac{352 - 22\lambda_j}{\lambda_j^2 - 59\lambda_j + 366} = \begin{cases} \sqrt{2} & , \quad j = 1 \\ 0 & , \quad j = 2 \\ -\sqrt{2} & , \quad j = 3 \end{cases} \quad (17b)$$

The modal masses become

$$M_j = \Phi^{(j)T} \mathbf{M} \Phi^{(j)} = \begin{cases} 4m & , \quad j = 1 \\ 2m & , \quad j = 2 \\ 4m & , \quad j = 3 \end{cases} \quad (18)$$

The displacement response is next expanded as a linear combination of the linear independent basis vectors $\Phi^{(1)}$, $\Phi^{(2)}$ and $\Phi^{(3)}$

$$\mathbf{x}(t) = q_1(t)\Phi^{(1)} + q_2(t)\Phi^{(2)} + q_3(t)\Phi^{(3)} \quad (19)$$

Using the decoupling condition $\Phi^{(i)T} \mathbf{C} \Phi^{(j)} = 0$, $i \neq j$, the differential equations for the modal coordinates are given by (3-187) with $F_i(t) \equiv 0$

$$\ddot{q}_j + 2\zeta_j \omega_j \dot{q}_j + \omega_j^2 q_j = 0, \quad t > 0, \quad j = 1, 2, 3 \quad (20)$$

The modal damping ratios follow from (3-182). Using (11) with $c = 0.32\sqrt{\frac{12}{7}}\sqrt{\frac{EI m}{l^3}}$ the following is obtained

$$\zeta_j = \frac{\Phi^{(j)T} \mathbf{C} \Phi^{(j)}}{2\omega_j M_j} = \frac{\frac{c}{m} \Phi^{(j)T} \mathbf{M} \Phi^{(j)}}{2\omega_j M_j} = \frac{c}{2\omega_j m} = \begin{cases} 0.02\sqrt{\frac{34+\sqrt{968}}{47}} & , \quad j = 1 \\ 0.01 & , \quad j = 2 \\ 0.02\sqrt{\frac{34-\sqrt{968}}{47}} & , \quad j = 3 \end{cases} \quad (21)$$

It follows that

$$\zeta_j \omega_j = \zeta_0 \omega_0 = \frac{c}{2m} = 0.16 \sqrt{\frac{12}{7}} \sqrt{\frac{EI}{ml^3}} \quad , \quad j = 1, 2, 3 \quad (22)$$

The initial values related to (20) follow from (3-175), (9), (10), (18)

$$q_j(0) = \frac{1}{M_j} \Phi^{(j)T} \mathbf{M} \mathbf{x}_0 = \begin{cases} (22 + 17\sqrt{2})q_0 & , \quad j = 1 \\ 0 & , \quad j = 2 \\ (22 - 17\sqrt{2})q_0 & , \quad j = 3 \end{cases} \quad (23)$$

$$\dot{q}_j(0) = \frac{1}{M_j} \Phi^{(j)T} \mathbf{M} \dot{\mathbf{x}}_0 = 0 \quad , \quad j = 1, 2, 3 \quad (24)$$

where

$$q_0 = \frac{7}{72192} \frac{f_0 l^3}{EI} \quad (25)$$

The solution to (20), (23), (24) follows from (3-188), (3-189)

$$q_j(t) = e^{-\zeta_0 \omega_0 t} \left(\cos(\omega_{d,j} t) + \frac{\zeta_j}{\sqrt{1 - \zeta_j^2}} \sin(\omega_{d,j} t) \right) q_j(0) \quad , \quad j = 1, 2, 3 \quad (26)$$

$$\omega_{d,j} = \omega_j \sqrt{1 - \zeta_j^2} \quad (27)$$

Because $q_2(0) = 0$ it follows that $q_2(t) \equiv 0$. The motion is then given by (17), (19)

$$\begin{aligned} \mathbf{x}(t) = q_0 e^{-\zeta_0 \omega_0 t} & \left(\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \left(\cos(\omega_{d,1} t) + \frac{\zeta_1}{\sqrt{1 - \zeta_1^2}} \sin(\omega_{d,1} t) \right) (22 + 17\sqrt{2}) + \right. \\ & \left. \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \left(\cos(\omega_{d,3} t) + \frac{\zeta_3}{\sqrt{1 - \zeta_3^2}} \sin(\omega_{d,3} t) \right) (22 - 17\sqrt{2}) \right) \end{aligned} \quad (28)$$

where $\zeta_0 \omega_0$ is given by (22) and q_0 is given by (25). The solution (28) has been depicted below in fig. 2.

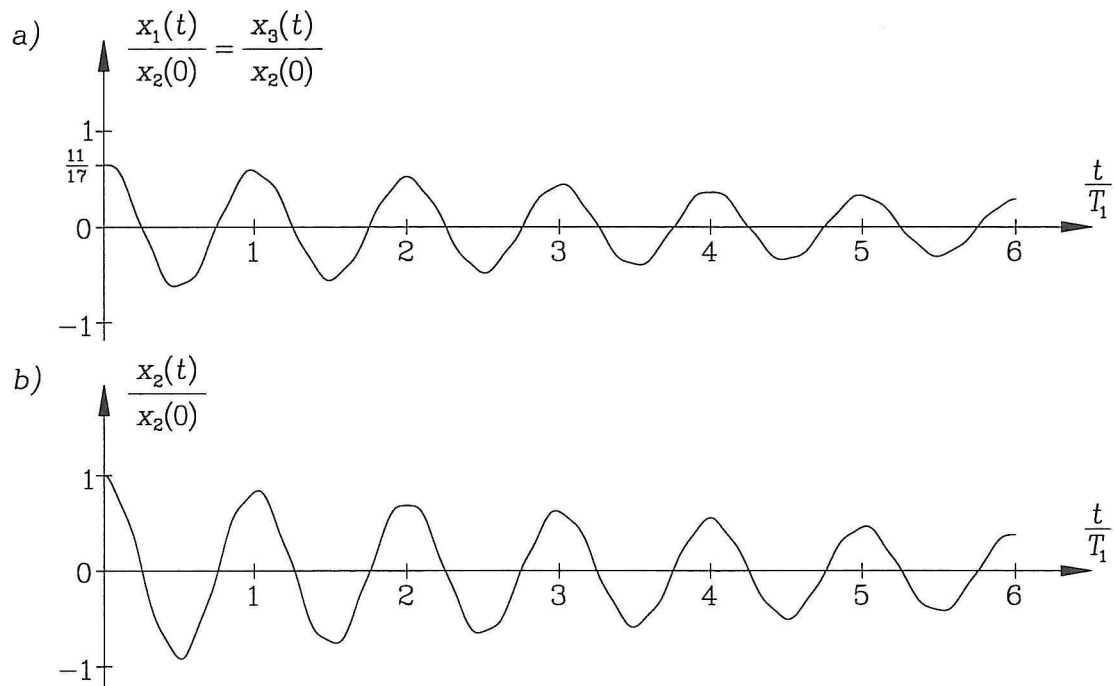


Fig. 2: Calculated eigenvibrations, $T_1 = \frac{2\pi}{\omega_1}$.

Note:

Since the structure is symmetric, and the loading is symmetric as well, the system could at once have been reduced to an equivalent 2 degrees-of-freedom system, with suitably selected boundary conditions in the symmetry section.

1.8 Lecture 8

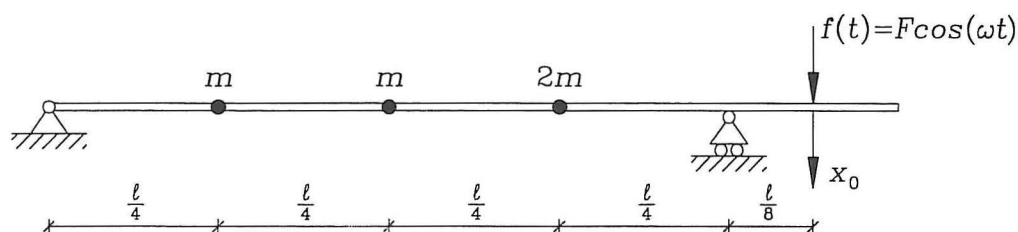
SUPPLEMENTARY PROBLEMS

- May 14, 1987. Problem 1, question 2.
- June 16, 1989. Problem 1, question 2.
- September 28, 1989. Problem 1.
- September 5, 1990. Problem 2, questions 2 and 3.
- June 21, 1991. Problem 1.
- June 21, 1991. Problem 2, question 2.
- September 4, 1991. Problem 1, question 2.
- September 4, 1991. Problem 2.
- June 12, 1992. Problem 1.
- June 16, 1993. Problem 2, question 2.
- June 21, 1996. Problem 2, question 2.
- August 26, 1996. Problem 2, question 3.
- June 22, 1998. Problem 3, question 3.

PROBLEM 1

June 20, 1988. Problem 1.

PROBLEM 2



The figure shows a homogeneous massless Bernoulli-Euler beam with the bending stiffness EI . The force on the cantilever part of the beam is assumed to have been acting for such a time that the response from any initial condition has been dissipated. Any influence from damping is ignored, and only small displacements from the initial state are considered.

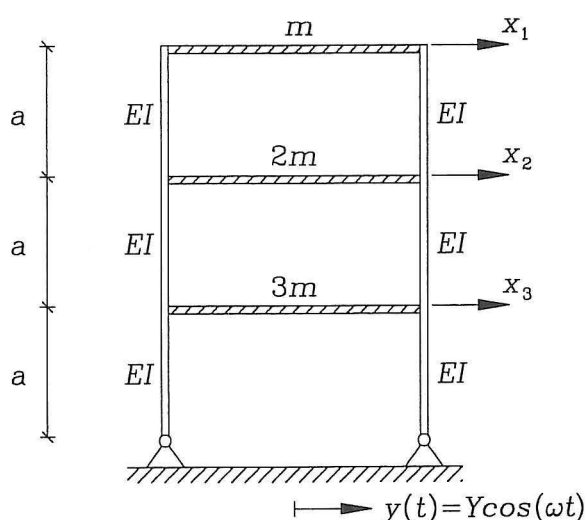
Question 1:

Determine the stationary response of each of the 3 masses as well as the point of action of the harmonic force with the amplitude F and the circular frequency ω .

Question 2:

Determine the circular frequencies $\omega = \omega_i'$ of the excitation for which the point of action of the harmonic force has the motion $x_0(t) \equiv 0$.

PROBLEM 3



All storey beams in the 3-storey plane frame shown above are assumed to be infinitely stiff. The masses of the beams have been indicated in the figure. All columns are homogeneously massless Bernoulli-Euler beams of the length a and the bending stiffness EI . The structure is excited by a harmonic varying displacement of the support, which is assumed to have been acting for such a time that the response from any initial conditions has been dissipated. Any influence from the damping is ignored, and only small displacements from the initial static equilibrium state are considered.

Question 1:

Determine the stationary response of each of the 3 storeys.

Question 2:

Determine the circular frequencies $\omega = \omega_i$ of the excitation for which the lowest storey has the motion $x_3(t) \equiv 0$.

SOLUTIONS

PROBLEM 1

See solution at June 20, 1988, problem 1.

PROBLEM 2

Question 1:

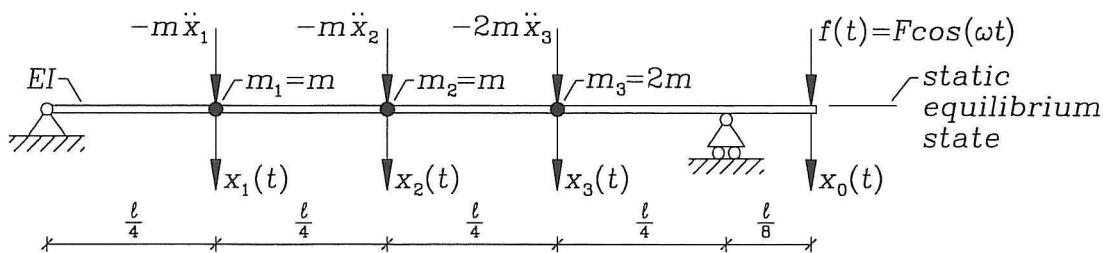


Fig. 1: Forces on the beam.

The beam is massless. Hence, the system has 3 degrees of freedom, which are selected as the vertical displacements $x_1(t)$, $x_2(t)$ and $x_3(t)$ of the masses from the static equilibrium state with signs as defined in fig. 1. The vertical displacement from the static equilibrium state of the point of action of the indirectly acting external force is denoted $x_0(t)$. The inertial forces $-m_i\ddot{x}_i(t)$ are applied as external forces and with signs as shown in fig. 1 according to d'Alembert's principle. The displacements $x_0(t)$ and $x_i(t)$, $i = 1, 2, 3$ are then given as, cf. (3-342), (3-343)

$$x_0(t) = \delta_{00}f(t) + \sum_{j=1}^3 \delta_{0j}(-m_j\ddot{x}_j) \quad (1)$$

$$x_i(t) = \delta_{i0}f(t) + \sum_{j=1}^3 \delta_{ij}(-m_j\ddot{x}_j), \quad i = 1, 2, 3 \quad (2)$$

The flexibility coefficients are given as, cf. (B-1), (B-3)

$$\delta_{00} = \frac{3}{512} \frac{\ell^3}{EI} \quad (3)$$

$$\mathbf{d}_0 = \begin{bmatrix} \delta_{01} \\ \delta_{02} \\ \delta_{03} \end{bmatrix} = \frac{1}{1024} \frac{\ell^3}{EI} \begin{bmatrix} -5 \\ -8 \\ -7 \end{bmatrix} \quad (4)$$

$$\mathbf{D} = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{12} & \delta_{22} & \delta_{23} \\ \delta_{13} & \delta_{23} & \delta_{33} \end{bmatrix} = \frac{1}{768} \frac{l^3}{EI} \begin{bmatrix} 9 & 11 & 7 \\ 11 & 16 & 11 \\ 7 & 11 & 9 \end{bmatrix} \quad (5)$$

The equations of motion become, cf. (3-345), (3-349)

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{K}\mathbf{d}_0 f(t) \quad (6)$$

$$x_0(t) = \delta_{00} f(t) - \mathbf{d}_0^T \mathbf{M}\ddot{\mathbf{x}} \quad (7)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix} \quad (8)$$

$$\mathbf{K} = \mathbf{D}^{-1} = \frac{192}{7} \frac{EI}{l^3} \begin{bmatrix} 23 & -22 & 9 \\ -22 & 32 & -22 \\ 9 & -22 & 23 \end{bmatrix} \quad (9)$$

With $f(t) = F \cos(\omega t)$ the stationary response of the masses is given as, cf. (3-352), (3-353)

$$\mathbf{x}(t) = \mathbf{X} \cos(\omega t) \quad (10)$$

$$\mathbf{X} = \mathbf{H}(\omega) \mathbf{F} \quad (11)$$

$$\mathbf{F} = \mathbf{K}\mathbf{d}_0 F = \frac{3}{56} \begin{bmatrix} -1 \\ 4 \\ -15 \end{bmatrix} F \quad (12)$$

$$\mathbf{H}(\omega) = (\mathbf{K} - \omega^2 \mathbf{M})^{-1} = \frac{1}{D} \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{bmatrix} \quad (13)$$

$$D = \frac{192}{7} \frac{EI}{l^3} (-2\lambda^3 + 133\lambda^3 - 1204\lambda + 784) \quad (13a)$$

$$h_{11} = 2\lambda^2 - 87\lambda + 252 \quad (13b)$$

$$h_{12} = -44\lambda + 308 \quad (13c)$$

$$h_{13} = 9\lambda + 196 \quad (13d)$$

$$h_{22} = 2\lambda^2 - 69\lambda + 448 \quad (13e)$$

$$h_{23} = -22\lambda + 308 \quad (13f)$$

$$h_{33} = \lambda^2 - 55\lambda + 252 \quad (13g)$$

where

$$\lambda = \frac{7}{192} \frac{\omega^2 m l^3}{EI} \quad (14)$$

$\mathbf{x}(t) = \mathbf{X} \cos(\omega t)$ is in phase with the excitation $f(t) = F \cos(\omega t)$, and \mathbf{X} is real, because the structural system is free of damping. From (11), (12), (13) it follows that

$$\mathbf{X} = \frac{1}{512(-2\lambda^3 + 133\lambda^2 - 1204\lambda + 784)} \begin{bmatrix} -2\lambda^2 - 224\lambda - 1960 \\ 8\lambda^2 + 98\lambda - 3136 \\ -15\lambda^2 + 728\lambda - 2744 \end{bmatrix} \frac{FL^3}{EI} \quad (15)$$

The stationary response of the point of action of the indirectly acting harmonic force is given as, cf. (3-354)

$$x_0(t) = X_0 \cos(\omega t) \quad (16)$$

where the amplitude X_0 follows from (7), (10)

$$X_0 = \delta_{00} F + \omega^2 \mathbf{d}_0^T \mathbf{M} \mathbf{X} \quad (17)$$

Inserting (3), (4), (8), (15), and eliminating ω^2 from (17) by means of (14), eq. (17) can be written

$$X_0 = \frac{3(-17\lambda^3 + 1260\lambda^2 - 15386\lambda + 21952)}{14336(-2\lambda^3 + 133\lambda^2 - 1204\lambda + 784)} \frac{Fl^3}{EI} \quad (18)$$

Question 2:

From (18) it follows that the displacement $x_0(t) \equiv 0$, if

$$-17\lambda^3 + 1260\lambda^2 - 15386\lambda + 21952 = 0 \quad \Rightarrow$$

$$\lambda'_j = \begin{cases} 1.6428866 & , \quad j = 1 \\ 13.277486 & , \quad j = 2 \\ 59.197274 & , \quad j = 3 \end{cases} \quad (19)$$

$$\omega'_j = \begin{cases} 6.713\sqrt{\frac{EI}{ml^3}} & , \quad j = 1 \\ 19.08\sqrt{\frac{EI}{ml^3}} & , \quad j = 2 \\ 40.30\sqrt{\frac{EI}{ml^3}} & , \quad j = 3 \end{cases} \quad (20)$$

Notes:

$\omega'_1, \omega'_2, \omega'_3$ signify the circular eigenfrequencies of the modified system in which a fixed support is supplied at the point of action of the force. Harmonic excitations of the unmodified system with these circular frequencies at even very small amplitudes X_0 will lead to significant displacements of the masses. This statement is also true in the limit as $X_0 \rightarrow 0$. This resolves the apparent paradox that non-zero displacements of the masses may take place even if $X_0 = 0$.

Instead of the applied direct evaluation of the frequency response matrix an expansion of this in terms of the eigenmodes as given by (3-353) can also be applied, i.e.

$$\mathbf{H}(\omega) = (\mathbf{K} - \omega^2 \mathbf{M})^{-1} = \sum_{j=1}^3 \frac{1}{M_j(\omega_j^2 - \omega^2)} \boldsymbol{\Phi}^{(j)} \boldsymbol{\Phi}^{(j)T} \quad (21)$$

PROBLEM 3

Question 1:

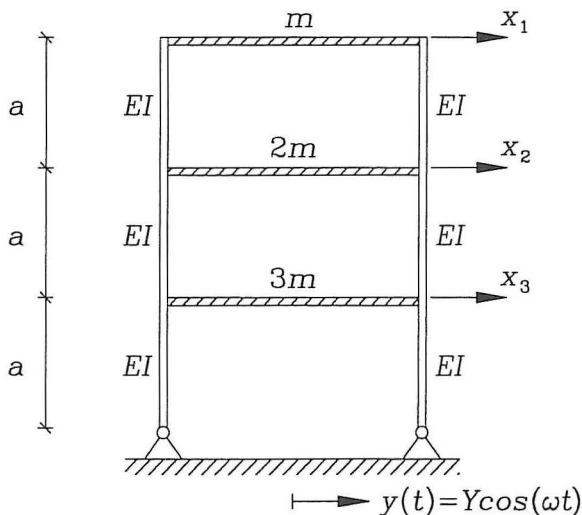


Fig. 1: 3-storey plane frame subjected to harmonic excitation of supports.

The frame has 3 degrees of freedom which are selected as shown in fig. 1. The equation of motion of the frame becomes, see (3-329)

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{K}\mathbf{U}y(t) \quad (1)$$

where, see (3-15)

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 3m \end{bmatrix} \quad (2)$$

$$\mathbf{K} = \frac{EI}{a^3} \begin{bmatrix} 24 & -24 & 0 \\ -24 & 48 & -24 \\ 0 & -24 & 30 \end{bmatrix} \quad (3)$$

\mathbf{U} is the influence vector specifying the quasi-static displacements in the selected degrees of freedom from unit displacement of the supports, $y(t) = 1$. Hence,

$$\mathbf{U} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (4)$$

With $y(t) = Y \cos(\omega t)$ the stationary response of the storeys is given as

$$\mathbf{x}(t) = \mathbf{X} \cos(\omega t) \quad (5)$$

$$\mathbf{X} = \mathbf{H}(\omega)\mathbf{F} \quad (6)$$

$$\mathbf{F} = \mathbf{K}\mathbf{U}Y = 6\frac{EI}{a^3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} Y \quad (7)$$

$$\mathbf{H}(\omega) = (\mathbf{K} - \omega^2\mathbf{M})^{-1} = \frac{1}{D} \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{bmatrix} \quad (8)$$

$$D = \frac{EI}{a^3} (-6\lambda^3 + 348\lambda^2 - 4032\lambda + 3456) \quad (8a)$$

$$h_{11} = 6\lambda^2 - 204\lambda + 864 \quad (8b)$$

$$h_{12} = -72\lambda + 720 \quad (8c)$$

$$h_{13} = 576 \quad (8d)$$

$$h_{22} = 3\lambda^2 - 102\lambda + 720 \quad (8e)$$

$$h_{23} = -24\lambda + 576 \quad (8f)$$

$$h_{33} = 2\lambda^2 - 96\lambda + 576 \quad (8g)$$

where

$$\lambda = \frac{\omega^2 m a^3}{EI} \quad (9)$$

$\mathbf{x}(t) = \mathbf{X} \cos(\omega t)$ is in phase with the excitation $y(t) = Y \cos(\omega t)$, and \mathbf{X} is real, because the structural system is free of damping. From (6), (7), (8) it follows that

$$\mathbf{X} = \frac{2}{-\lambda^3 + 58\lambda^2 - 672\lambda + 576} \begin{bmatrix} 288 \\ 288 - 12\lambda \\ 288 - 48\lambda + \lambda^2 \end{bmatrix} \frac{EI}{a^3} Y \quad (10)$$

Question 2:

From (10) it follows that the lowest storey has the displacement $x_3(t) \equiv 0$, if

$$\lambda^2 - 48\lambda + 288 = 0 \Rightarrow$$

$$\lambda'_j = \begin{cases} 24 - \sqrt{288} & , \quad j = 1 \\ 24 + \sqrt{288} & , \quad j = 2 \end{cases} \quad (11)$$

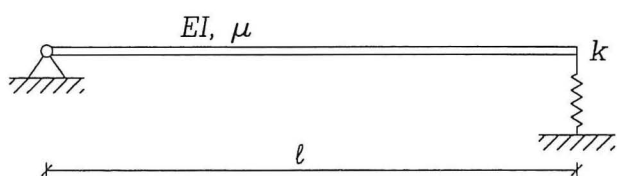
$$\omega'_j = \begin{cases} \sqrt{24 - \sqrt{288}} \sqrt{\frac{EI}{ma^3}} & , \quad j = 1 \\ \sqrt{24 + \sqrt{288}} \sqrt{\frac{EI}{ma^3}} & , \quad j = 2 \end{cases} \quad (12)$$

1.9 Lecture 9

SUPPLEMENTARY PROBLEMS

- May 14, 1987. Problem 2.
- June 20, 1988. Problem 2.
- September 5, 1990. Problem 3.
- June 21, 1991. Problem 3.
- September 4, 1991. Problem 3.
- June 12, 1992. Problem 3, question 1.
- June 22, 1994. Problem 3.
- August 24, 1994. Problem 4.
- June 28, 1995. Problem 3.
- June 21, 1996. Problem 3.
- August 26, 1996. Problem 3.
- June 19, 1997. Problem 2.
- June 22, 1998. Problem 2.

PROBLEM 1



The figure shows a plane homogeneous Bernoulli-Euler beam of the length l . The bending stiffness is EI and the mass per unit length is μ . At one end section the beam is fixed and simply supported. The other end section is supported by a linear elastic spring with the spring constant k . Only small vibrations from the static equilibrium state are considered.

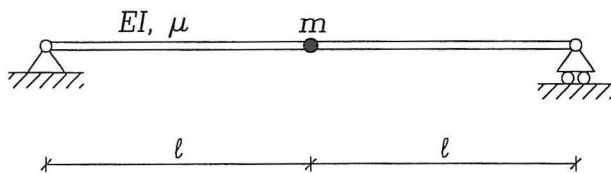
Question 1:

Formulate the frequency condition for the determination of the undamped circular eigenfrequencies of the system.

Question 2:

Determine the 5 lowest circular eigenfrequencies of the system for $k = 10 \frac{EI}{l^3}$, and compare the result with the corresponding circular eigenfrequencies of a simply supported beam.

PROBLEM 2



The figure shows a plane homogeneous Bernoulli-Euler beam of the length $2l$. The bending stiffness is EI and the mass per unit length is μ . The beam is simply supported at both end sections. At the mid-section a point mass m is attached. Only small vibrations from the static equilibrium state are considered.

Question 1:

Formulate the frequency condition for the determination of the undamped circular eigenfrequencies of the system.

Question 2:

Determine the 5 lowest circular eigenfrequencies of the system for $m = \mu l$.

PROBLEM 3

June 16, 1989. Problem 1, question 3.

SOLUTIONS

PROBLEM 1

Question 1:

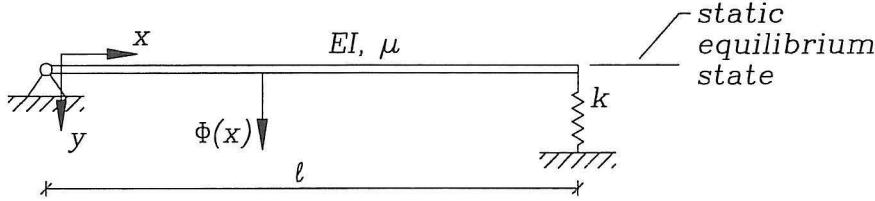


Fig. 1: Beam with constant bending stiffness and constant mass per unit length.

The beam has constant bending stiffness EI and constant mass per unit length μ . Further, it is assumed that the normal force is $N = 0$. Hence, the eigenmodes are given by (4-18), (4-19)

$$\Phi(x) = A \sin\left(\lambda \frac{x}{l}\right) + B \cos\left(\lambda \frac{x}{l}\right) + C \sinh\left(\lambda \frac{x}{l}\right) + D \cosh\left(\lambda \frac{x}{l}\right) \quad (1)$$

$$\lambda^4 = \frac{\mu \omega^2 l^4}{EI} \quad (2)$$

The boundary conditions at $x = 0$ become, see (4-23)

$$\Phi(0) = \frac{d^2}{dx^2} \Phi(0) = 0 \quad (3)$$

(3) implies that $B = D = 0$, see (4-24). Then, (1) reduces to

$$\Phi(x) = A \sin\left(\lambda \frac{x}{l}\right) + C \sinh\left(\lambda \frac{x}{l}\right) \quad (4)$$

The mechanical boundary conditions at $x = l$ become, see (4-13)

$$\frac{d^2}{dx^2} \Phi(l) = 0 \quad (5)$$

$$EI \frac{d^3}{dx^3} \Phi(l) = k \Phi(l) \quad (6)$$

Insertion of (4) into (5) and (6) provides the following homogeneous equations for the determination of A and C

$$\left. \begin{aligned} \frac{\lambda^2}{l^2}(-A \sin \lambda + C \sinh \lambda) &= 0 \\ EI \frac{\lambda^3}{l^3}(-A \cos \lambda + C \cosh \lambda) &= k(A \sin \lambda + C \sinh \lambda) \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} -\sin \lambda & \sinh \lambda \\ -\lambda^3 \cos \lambda - \kappa \sin \lambda & \lambda^3 \cosh \lambda - \kappa \sinh \lambda \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (7)$$

where

$$\kappa = \frac{kl^3}{EI} \quad (8)$$

The eigenmode (4) is non-trivial if $A \neq 0 \vee C \neq 0$. Solutions $A \neq 0 \vee C \neq 0$ to (7) can only be found, if the determinant of the coefficient matrix is equal to zero. The frequency condition then becomes

$$\begin{aligned} -\sin \lambda (\lambda^3 \cosh \lambda - \kappa \sinh \lambda) + \sinh \lambda (\lambda^3 \cos \lambda + \kappa \sin \lambda) &= 0 \Rightarrow \\ \lambda^3 \cot \lambda - \lambda^3 \coth \lambda + 2\kappa &= 0 \end{aligned} \quad (9)$$

Question 2:

$k = 10 \frac{EI}{l^3} \Rightarrow \kappa = 10$. Inserting $\kappa = 10$ into (9), the following solutions for the frequency parameter λ are obtained by means of a suitable iteration scheme

$$\lambda_j = \begin{cases} 2.23133 & , \quad j = 1 \\ 4.09539 & , \quad j = 2 \\ 7.09735 & , \quad j = 3 \\ 10.21963 & , \quad j = 4 \\ 13.35598 & , \quad j = 5 \end{cases} \quad (10)$$

The corresponding circular eigenfrequencies follow from (2)

$$\omega_j = \begin{cases} 4.9788 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 1 \\ 16.7722 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 2 \\ 50.3724 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 3 \\ 104.4408 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 4 \\ 178.3822 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 5 \end{cases} \quad (11)$$

The circular eigenfrequencies for a simply supported beam are given by (4-33)

$$\omega_j = \begin{cases} 9.8696 \sqrt{\frac{EI}{\mu l^4}} , & j = 1 \\ 39.4784 \sqrt{\frac{EI}{\mu l^4}} , & j = 2 \\ 88.8264 \sqrt{\frac{EI}{\mu l^4}} , & j = 3 \\ 157.9137 \sqrt{\frac{EI}{\mu l^4}} , & j = 4 \\ 246.7401 \sqrt{\frac{EI}{\mu l^4}} , & j = 5 \end{cases} \quad (12)$$

As seen, the circular eigenfrequencies (11) are significantly smaller than the corresponding circular eigenfrequencies (12). From this it is concluded that $k = 10 \frac{EI}{l^3}$ is a relatively soft spring. Further, it is seen that the effect of the spring is somewhat more sensible to the lowest eigenfrequency.

PROBLEM 2

Question 1:

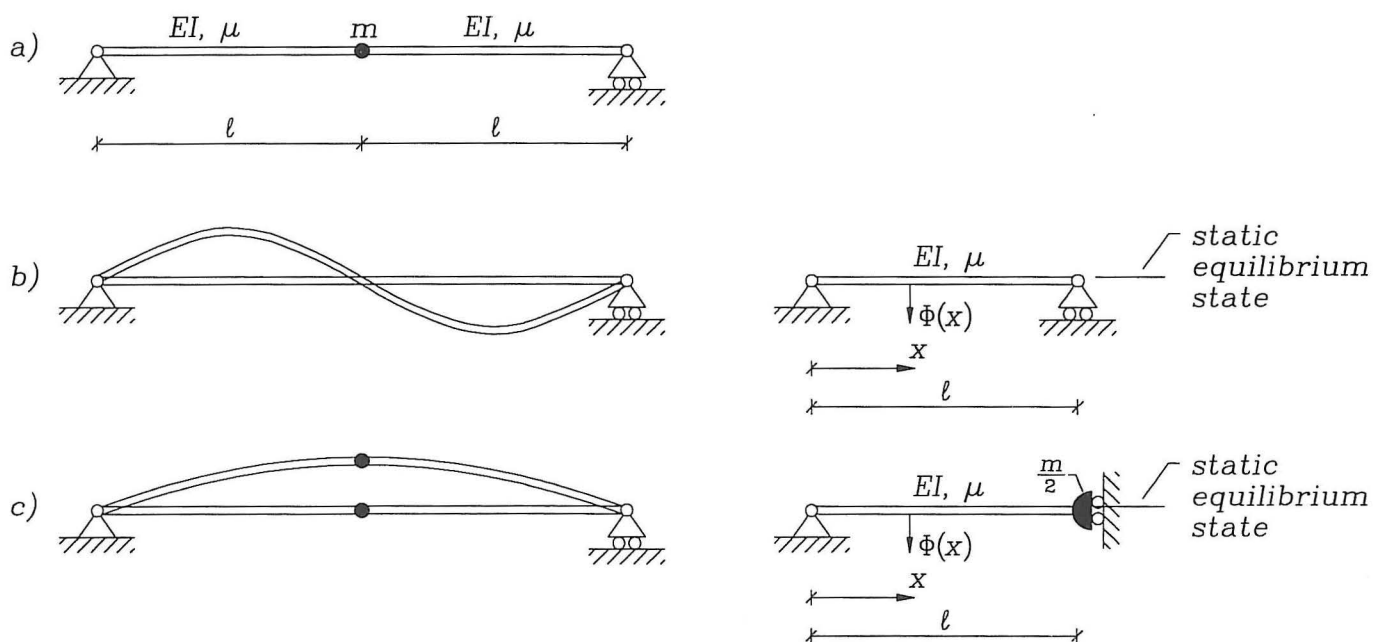


Fig. 1: a) Symmetric structural system. b) Anti-symmetric eigenvibrations and equivalent system. c) Symmetric eigenvibrations and equivalent system.

The system is mechanically and geometrically symmetric around the mid-point. Hence, the eigenvibrations separate into the anti-symmetric and symmetric eigenvibrations as shown in the figs. 1b and 1c.

Anti-symmetric eigenvibrations are characterized by zero displacements at the symmetry point. Further, the curvature and hence the moment is zero at the symmetry point. For this reason anti-symmetric eigenvibrations can be analysed by the equivalent reduced system as shown in fig. 1b, which is a simply supported homogeneous beam of the length l . Hence, anti-symmetric circular eigenfrequencies become, cf. (4-33)

$$\omega_{2j} = j^2 \pi^2 \sqrt{\frac{EI}{\mu l^4}} \quad , \quad j = 1, 2, \dots \quad (1)$$

Symmetric eigenvibrations are characterized by the zero slope at the symmetry point. In the "middle" of the mass m the shear force is zero. For this reason symmetric eigenvibrations can be analysed by the reduced system shown in fig. 1c. The eigenmode of the symmetric eigenvibrations is then the solution to the linear eigenvalue problem, cf. (4-13)

$$\left. \begin{aligned} EI \frac{d^4 \Phi}{dx^4} - \mu \omega^2 \Phi &= 0 \\ \Phi(0) &= 0 \\ \frac{d}{dx} \Phi(l) &= 0 \\ \frac{d^2}{dx^2} \Phi(0) &= 0 \\ EI \frac{d^3}{dx^3} \Phi(l) &= -\omega^2 \frac{m}{2} \Phi(l) \end{aligned} \right\} \quad (2)$$

The solution of (2) is given by (4-18), (4-19)

$$\Phi(x) = A \sin \left(\lambda \frac{x}{l} \right) + B \cos \left(\lambda \frac{x}{l} \right) + C \sinh \left(\lambda \frac{x}{l} \right) + D \cosh \left(\lambda \frac{x}{l} \right) \quad (3)$$

$$\lambda^4 = \frac{\mu \omega^2 l^4}{EI} \quad (4)$$

The boundary conditions at $x = 0$ imply $B = D = 0$, see (4-24). Then (3) reduces to

$$\Phi(x) = A \sin \left(\lambda \frac{x}{l} \right) + C \sinh \left(\lambda \frac{x}{l} \right) \quad (5)$$

Inserting (5) into the boundary conditions at $x = l$ the following homogeneous equations are obtained for the determination of A and C

$$\left. \begin{aligned} \frac{\lambda}{l} (A \cos \lambda + C \cosh \lambda) &= 0 \\ EI \frac{\lambda^3}{l^3} (-A \cos \lambda + C \cosh \lambda) &= -\omega^2 \frac{m}{2} (A \sin \lambda + C \sinh \lambda) \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} \cos \lambda & \cosh \lambda \\ -\cos \lambda + \gamma \lambda \sin \lambda & \cosh \lambda + \gamma \lambda \sinh \lambda \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6)$$

where

$$\gamma = \frac{m}{2\mu l} \quad (7)$$

The frequency condition then becomes

$$\begin{aligned} \cos \lambda (\cosh \lambda + \gamma \lambda \sinh \lambda) - \cosh \lambda (-\cos \lambda + \gamma \lambda \sin \lambda) &= 0 \Rightarrow \\ 2 + \gamma \lambda (\tanh \lambda - \tan \lambda) &= 0 \end{aligned} \quad (8)$$

Question 2:

$m = \mu l \Rightarrow \gamma = \frac{1}{2}$. Inserting $\gamma = \frac{1}{2}$ into (8), the following solutions for the frequency parameter λ is obtained by means of a suitable iteration scheme

$$\lambda_j = \begin{cases} 1.31966 & , \quad j = 1 \\ 4.23720 & , \quad j = 3 \\ 7.28084 & , \quad j = 5 \\ 10.37046 & , \quad j = 7 \\ 13.48025 & , \quad j = 9 \end{cases} \quad (9)$$

The corresponding circular eigenfrequencies follow from (4)

$$\omega_j = \begin{cases} 1.7415 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 1 \\ 17.9539 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 3 \\ 53.0106 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 5 \\ 107.5464 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 7 \\ 181.7171 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 9 \end{cases} \quad (10)$$

As seen from (1) and (10) anti-symmetric and symmetric eigenvibrations are changing, and the lowest mode is symmetric.

PROBLEM 3

See solution at June 16, 1989. Problem 1, question 3.

1.10 Lecture 10

SUPPLEMENTARY PROBLEMS

- June 28, 1995. Problem 2.
- June 21, 1996. Problem 4.
- August 26, 1996. Problem 4.

PROBLEM 1

June 19, 1997. Problem 3.

PROBLEM 2

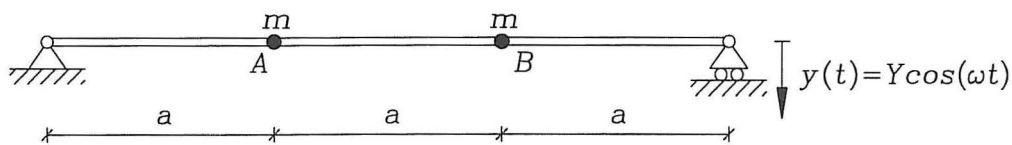
June 22, 1998. Problem 3, question 1.

2. PROBLEMS FROM WRITTEN EXAMINATIONS

2.1 May 14, 1987

Duration: 3h

PROBLEM 1



The figure shows a plane horizontal massless linear elastic Bernoulli-Euler beam, free of damping and of the length $3a$. The beam has constant bending stiffness EI and is simply supported at the end sections. At the third points A and B of the beam point masses of magnitude m have been attached. The structure is excited by a harmonic displacement $y(t) = Y \cos(\omega t)$ at the right support. Only small vibrations from the static equilibrium state in the vertical direction are considered.

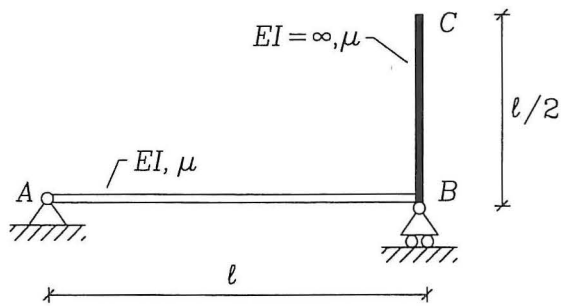
Question 1 (20%. $\mu = 16.3\%$)

Determine the undamped circular eigenfrequencies, the eigenmodes and the modal masses of the structure.

Question 2 (13%. $\mu = 7.2\%$)

Determine the stationary response of the system, when any influence from the initial conditions has been dissipated.

PROBLEM 2



The figure shows a plane beam system consisting of two sub-beams AB and BC . Beam AB is a linear elastic Bernoulli-Euler beam of the length l with constant bending stiffness EI and constant mass per unit length μ . Beam BC has the length $\frac{l}{2}$, and likewise constant mass per unit length μ . However, the beam is assumed to be infinitely stiff in bending and axial deformations. Only small vibrations of the beam AB in the transverse direction are considered.

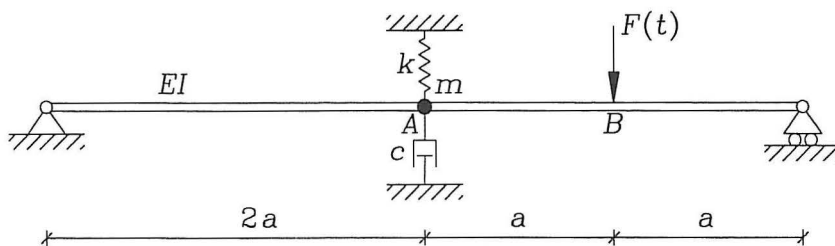
Question 1 (25%. $\mu = 5.9\%$)

Formulate the frequency condition for the determination of the undamped circular eigenfrequencies at eigenvibrations in the plane of the structure.

Question 2 (8%. $\mu = 6.8\%$)

Sketch the lowest eigenmode (no calculations are requested).

PROBLEM 3



The figure shows a plane horizontal massless linear elastic Bernoulli-Euler beam free of damping. The beam length is $4a$, the bending stiffness is constantly equal to EI , and the beam is simply supported at the end sections. At the mid-point A a point mass m , a linear viscous damper with the damping constant c , and a linear elastic spring with the spring constant k have been applied. At the fourth point B the structure is excited by an external time-varying load $F(t)$ acting in the vertical direction. Only small vibrations from the static equilibrium state in the vertical direction are considered.

Question 1 (20%. $\mu = 16.9\%$)

Determine the equation of motion of the structure and determine the undamped circular eigenfrequency and the damping ratio.

SOLUTIONS

PROBLEM 1

Question 1:

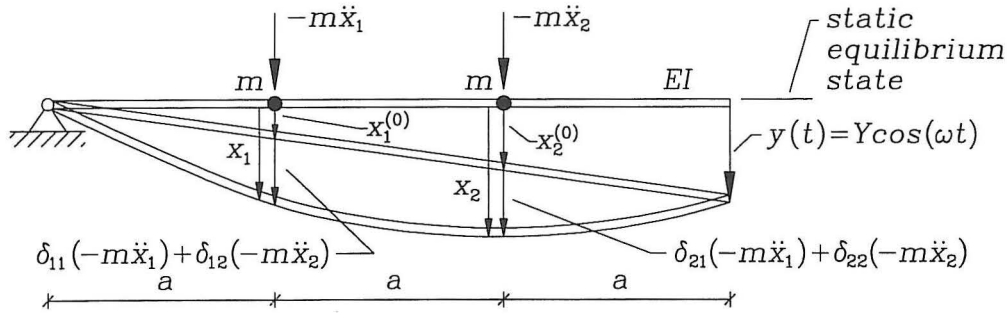


Fig. 1: Oscillating beam system.

The beam is massless. Hence, the system has but 2 degrees of freedom, which are selected as the vertical displacements $x_1(t)$ and $x_2(t)$ of the point masses from the static equilibrium state with signs as defined in fig. 1. The displacement $x_i(t)$ is made up of a quasi-static component $x_i^{(0)}(t)$ from the stiff body motion of the beam and a dynamic component $\delta_{i1}(-m\ddot{x}_1) + \delta_{i2}(-m\ddot{x}_2)$ from the inertial forces $-m\ddot{x}_1$ and $-m\ddot{x}_2$. The quasi-static motion can be written

$$\mathbf{x}^{(0)}(t) = \begin{bmatrix} x_1^{(0)}(t) \\ x_2^{(0)}(t) \end{bmatrix} = \mathbf{U}y(t) \quad , \quad \mathbf{U} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \quad (1)$$

The equations of motion then become, cf. (3-328)

$$\left. \begin{aligned} x_1(t) &= x_1^{(0)}(t) + \delta_{11}(-m\ddot{x}_1) + \delta_{12}(-m\ddot{x}_2) \\ x_2(t) &= x_2^{(0)}(t) + \delta_{21}(-m\ddot{x}_1) + \delta_{22}(-m\ddot{x}_2) \end{aligned} \right\} \quad (2)$$

The flexibility coefficients are given as, see (B-1)

$$\mathbf{D} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \frac{a^3}{18EI} \begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix} \quad (3)$$

The equations of motion (2) can then be written, cf. (3-329)

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{K}\mathbf{U}y(t) \quad (4)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad \mathbf{K} = \frac{6EI}{5a^3} \begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix} \quad (5)$$

The circular eigenfrequencies ω_j and the eigenmodes $\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix}$ are obtained as non-trivial solutions of the homogeneous linear equations, cf. (3-42)

$$\begin{bmatrix} 8 - \lambda_j & -7 \\ -7 & 8 - \lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6)$$

$$\lambda_j = \frac{5}{6} \frac{\omega_j^2 m a^3}{EI} \quad (7)$$

The characteristic equation becomes

$$(8 - \lambda_j)^2 - 7^2 = 0 \Rightarrow$$

$$\lambda_j = \begin{cases} 1 & , \quad j = 1 \\ 15 & , \quad j = 2 \end{cases} \quad (8)$$

$$\omega_j = \begin{cases} \sqrt{\frac{6}{5}} \sqrt{\frac{EI}{ma^3}} & , \quad j = 1 \\ \sqrt{18} \sqrt{\frac{EI}{ma^3}} & , \quad j = 2 \end{cases} \quad (9)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix} \quad (10)$$

The first component $\Phi_1^{(j)}$ is determined from the first equation of (6) as follows

$$(8 - \lambda_j) \Phi_1^{(j)} - 7 \cdot 1 = 0 \Rightarrow$$

$$\Phi_1^{(j)} = \frac{7}{8 - \lambda_j} = \begin{cases} 1 & , \quad j = 1 \\ -1 & , \quad j = 2 \end{cases} \quad (11)$$

The modal masses become, see (3-152)

$$M_j = \Phi^{(j)T} \mathbf{M} \Phi^{(j)} = 2m \quad , \quad j = 1, 2 \quad (12)$$

Question 2:

With $y(t) = Y \cos(\omega t)$ the stationary response of the masses are given as

$$x(t) = \mathbf{X} \cos(\omega t) \quad (13)$$

$$\mathbf{X} = \mathbf{H}(\omega) \mathbf{F} \quad (14)$$

$$\mathbf{F} = \mathbf{K} \mathbf{U} Y = \frac{6}{5} \frac{EI}{a^3} \begin{bmatrix} -2 \\ 3 \end{bmatrix} Y \quad (15)$$

$$\mathbf{H}(\omega) = (\mathbf{K} - \omega^2 \mathbf{M})^{-1} = \sum_{j=1}^2 \frac{\boldsymbol{\Phi}^{(j)} \boldsymbol{\Phi}^{(j)T}}{M_j (\omega_j^2 - \omega^2)} \quad (16)$$

Inserting (11), (12), (15), (16) in (14) the following solution is obtained

$$\begin{aligned} \mathbf{X} &= \frac{3}{5} \left(\frac{1}{\omega_1^2 - \omega^2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{5}{\omega_2^2 - \omega^2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) \frac{EI}{ma^3} Y = \\ &= \frac{1}{2} \left(\frac{\omega_1^2}{\omega_1^2 - \omega^2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3} \frac{\omega_2^2}{\omega_2^2 - \omega^2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) Y \end{aligned} \quad (17)$$

where ω_1^2 and ω_2^2 are given by (9).

$\mathbf{x}(t) = \mathbf{X} \cos(\omega t)$ is in phase with the excitation $y(t) = Y \cos(\omega t)$ and \mathbf{X} is real, because the structural system is free of damping. Instead of the applied evaluation of the frequency response matrix in terms of the eigenmodes, a straightforward evaluation of $\mathbf{H}(\omega) = (\mathbf{K} - \omega^2 \mathbf{M})^{-1}$ can alternatively be performed in the present case of a 2 degrees-of-freedom system.

PROBLEM 2

Question 1:

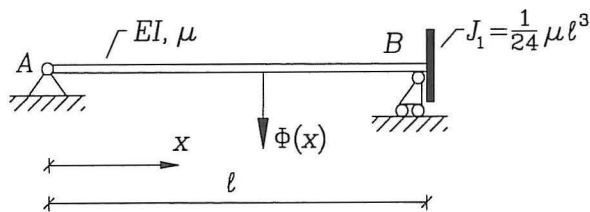


Fig. 1: Equivalent structural system.

The beam BC is equivalent to a distributed mass at point B with the mass moment of inertia

$$J_1 = \frac{1}{3}\mu \left(\frac{l}{2}\right)^3 = \frac{1}{24}\mu l^3 \quad (1)$$

The beam AB has constant bending stiffness EI and constant mass per unit length μ . Hence, the eigenmodes are given by (4-18), (4-19)

$$\Phi(x) = A \sin\left(\lambda \frac{x}{l}\right) + B \cos\left(\lambda \frac{x}{l}\right) + C \sinh\left(\lambda \frac{x}{l}\right) + D \cosh\left(\lambda \frac{x}{l}\right) \quad (2)$$

$$\lambda^4 = \frac{\mu\omega^2 l^4}{EI} \quad (3)$$

The boundary conditions at $x = 0$ become, see (4-23)

$$\Phi(0) = \frac{d^2}{dx^2}\Phi(0) = 0 \quad (4)$$

(4) implies that $B = D = 0$, see (4-24). Then (2) reduces to

$$\Phi(x) = A \sin\left(\lambda \frac{x}{l}\right) + C \sinh\left(\lambda \frac{x}{l}\right) \quad (5)$$

The boundary conditions at $x = l$ become, see (4-13)

$$\Phi(l) = 0 \quad (6)$$

$$EI \frac{d^2}{dx^2}\Phi(l) = \omega^2 J_1 \frac{d}{dx}\Phi(l) \quad (7)$$

Inserting (5) into (6) and (7) the following homogeneous equations are obtained for the determination of A and C

$$\left. \begin{aligned} A \sin \lambda + C \sinh \lambda &= 0 \\ EI \frac{\lambda^2}{l^2} (-A \sin \lambda + C \sinh \lambda) &= \frac{\lambda}{l} \omega^2 J_1 (A \cos \lambda + C \cosh \lambda) \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} \sin \lambda & \sinh \lambda \\ \sin \lambda + \frac{\lambda^3}{24} \cos \lambda & -\sinh \lambda + \frac{\lambda^3}{24} \cosh \lambda \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (8)$$

where $\omega^2 J_1 = \frac{1}{24}\lambda^4 \frac{EI}{l}$ has been introduced into the 2nd equation, see (1), (3). Non-trivial solutions of (8) are obtained if, and only if the determinant of the coefficient matrix becomes equal to zero. The frequency condition then becomes

$$\sin \lambda \left(-\sinh \lambda + \frac{\lambda^3}{24} \cosh \lambda \right) - \sinh \lambda \left(\sin \lambda + \frac{\lambda^3}{24} \cos \lambda \right) = 0 \Rightarrow$$

$$\tan \lambda \tanh \lambda = \frac{\lambda^3}{48} (\tan \lambda - \tanh \lambda) \quad (9)$$

The 5 lowest solutions of (9) and the corresponding circular eigenfrequencies become

$$\lambda_j = \begin{cases} 2.60833 & , \quad j = 1 \\ 4.32561 & , \quad j = 2 \\ 7.13908 & , \quad j = 3 \\ 10.23308 & , \quad j = 4 \\ 13.36193 & , \quad j = 5 \end{cases} \quad (10)$$

$$\omega_j = \begin{cases} 6.8034 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 1 \\ 18.7109 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 2 \\ 50.9665 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 3 \\ 104.7159 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 4 \\ 178.5412 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 5 \end{cases} \quad (11)$$

Question 2:

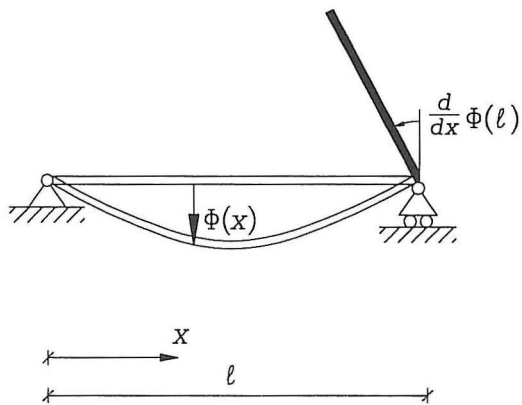


Fig. 2: Sketch of lowest eigenmode.

PROBLEM 3

Question 1:

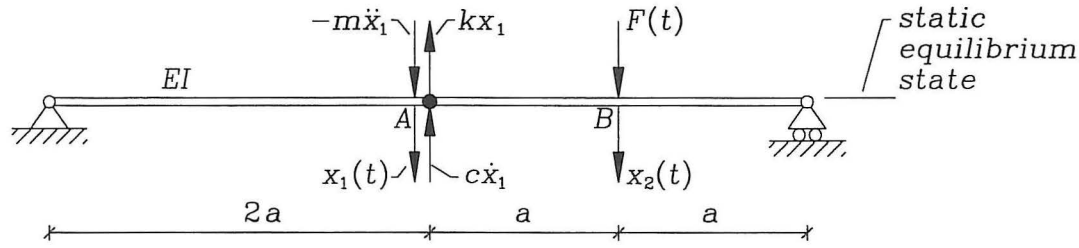


Fig. 1: Forces on a free beam.

Because the beam is massless, the system has but a single degree of freedom, which is selected as the vertical displacement $x_1(t)$ of point A from the static equilibrium state with a sign as defined in the figure. Besides, an artificial degree of freedom $x_2(t)$ indicating the vertical displacement of point B is introduced. The beam is cut free from the spring and from the damper, and the spring force $kx_1(t)$ and the damper force $c\dot{x}_1(t)$ are applied as external forces with signs as shown in fig. 1. Further, the inertial load $-m\ddot{x}_1$ is applied as an external load acting on the mass according to d'Alembert's principle. The equation of motion then reads, cf. (2-160)

$$x_1(t) = \delta_{11}(-m\ddot{x}_1 - c\dot{x}_1 - kx_1) + \delta_{12}F(t) \Rightarrow$$

$$m\ddot{x}_1 + c\dot{x}_1 + \left(k + \frac{1}{\delta_{11}}\right)x_1 = \frac{\delta_{12}}{\delta_{11}}F(t) \quad (1)$$

The flexibility coefficients become, cf. (B-1)

$$\delta_{11} = \frac{4}{3} \frac{a^3}{EI}, \quad \delta_{12} = \frac{11}{12} \frac{a^3}{EI} \quad (12)$$

Eq. (1) can then be written

$$m(\ddot{x}_1 + 2\zeta\omega_0\dot{x}_1 + \omega_0^2x_1) = \frac{11}{16}F(t) = X(t) \quad (3)$$

where the circular eigenfrequency ω_0 and the damping ratio ζ are given as, cf. (2-7), (2-39)

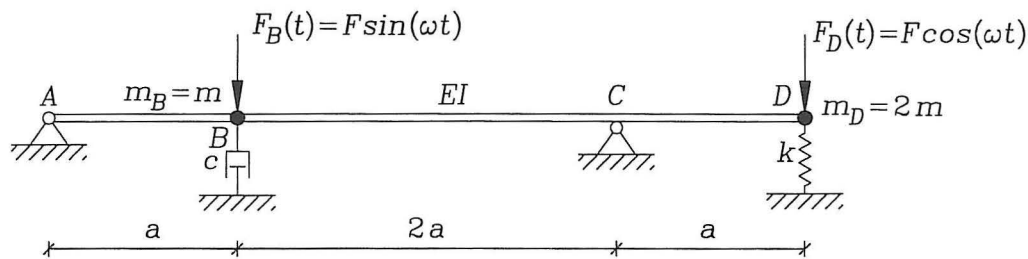
$$\omega_0 = \sqrt{\frac{k + \frac{3}{4} \frac{EI}{a^3}}{m}} \quad (4)$$

$$\zeta = \frac{c}{2m\omega_0} = \frac{c}{\sqrt{m(4k + 3 \frac{EI}{a^3})}} \quad (5)$$

2.2 June 26, 1987

Duration: 3h

PROBLEM 1



The figure shows a plane horizontal massless linear elastic Bernoulli-Euler beam of the length $4a$ and free of damping. The beam has constant bending stiffness EI , and is simply supported at one end section at point A . At the other end section at point D the beam is supported by a linear elastic spring with the spring constant k . At the quarter point C at the distance a from point D , a simple support is present. At quarter point B at the distance a from point A a linear viscous damper with the damping constant c is attached. At the points B and D point masses of magnitudes $m_B = m$ and $m_D = 2m$ are applied. At these points the structure is subjected to external dynamic loads $F_B(t) = F \sin(\omega t)$ and $F_D(t) = F \cos(\omega t)$ acting in the vertical direction. Only small vibrations from the static equilibrium state in the vertical direction are considered.

Question 1 (15%. $\mu = 11.7\%$)

Determine the equations of motion of the structure.

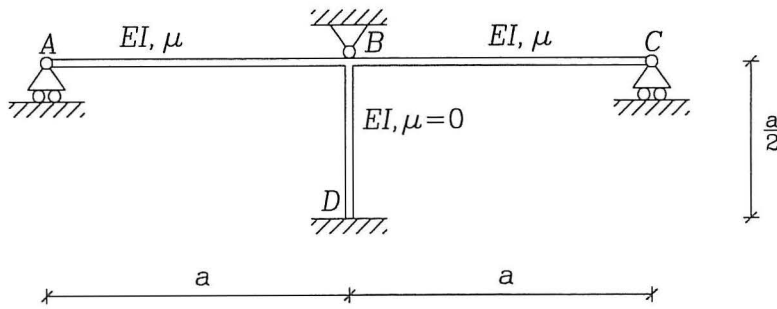
Question 2 (10%. $\mu = 5.7\%$)

Determine the undamped circular eigenfrequencies and eigenmodes of the structure for $k = \frac{9}{8} \frac{EI}{a^3}$.

Question 3 (15%. $\mu = 4.3\%$)

Determine the stationary displacement response of point B and point D when any influence from the initial conditions has been dissipated.

PROBLEM 2



The figure shows a plane beam system consisting of the beams AB , BC and BD . The beams AB and BC are horizontal linear elastic Bernoulli-Euler beams of the length a with constant bending stiffness EI and constant mass per unit length μ . The beam BD is a vertical massless ($\mu = 0$) linear elastic Bernoulli-Euler beam of the length $\frac{a}{2}$, also with constant bending stiffness EI . All beams are assumed to be infinitely stiff against axial elongations, and are fixed to each other at point B . At the points A and C a movable simple support is attached. Horizontal movements of the beam ABC are prevented by a fixed single support at point B . Beam BD is fixed at point D . Only small vibrations from the static equilibrium state in the transverse direction of the beams are considered. Influence from any damping mechanism is ignored.

Question 1 (30%. $\mu = 10\%$)

Formulate the frequency condition for the determination of the undamped circular eigenfrequencies of the structure.

SOLUTIONS

PROBLEM 1

Question 1:

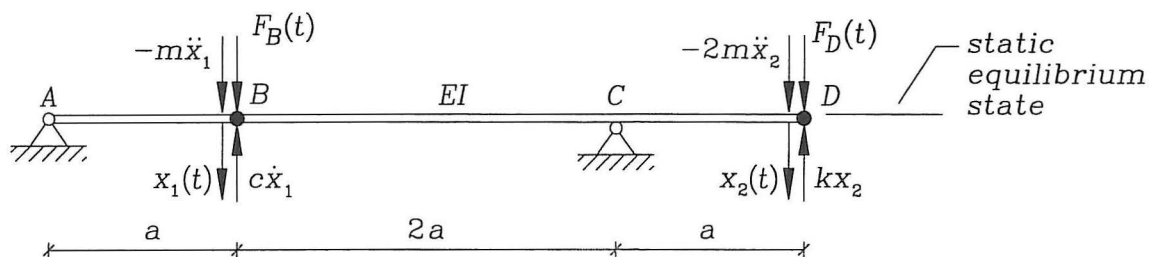


Fig. 1: Forces on a free beam.

The beam is massless. Hence, the system has but 2 degrees of freedom, which are selected as the vertical displacements $x_1(t)$ and $x_2(t)$ of the points B and D from the static equilibrium state with signs as defined in fig. 1. The beam is cut free from the damper at point B and from the spring at point D , and the damper force $c\dot{x}_1(t)$ and the spring force $kx_2(t)$ are applied as external forces with signs as shown in fig. 1. Further, the inertial loads $-m\ddot{x}_1(t)$ and $-2m\ddot{x}_2(t)$ are applied as external loads acting at points B and D according to d'Alembert's principle. The equation of motion then reads, cf. (3-1)

$$\left. \begin{aligned} x_1(t) &= \delta_{11}(F_B(t) - m\ddot{x}_1 - c\dot{x}_1) + \delta_{12}(F_D(t) - 2m\ddot{x}_2 - kx_2) \\ x_2(t) &= \delta_{21}(F_B(t) - m\ddot{x}_1 - c\dot{x}_1) + \delta_{22}(F_D(t) - 2m\ddot{x}_2 - kx_2) \end{aligned} \right\} \quad (1)$$

The flexibility coefficients are given as, see (B-1), (B-2), (B-3)

$$\mathbf{D} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \frac{4}{9} \frac{a^3}{EI} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \quad (2)$$

Using $k = \frac{9}{8} \frac{EI}{a^3}$, the equations of motion (2) can then be written in the following matrix form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \quad (3)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \quad (4)$$

$$\mathbf{K} = \mathbf{D}^{-1} + \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix} = \frac{9}{8} \frac{EI}{a^3} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \quad (5)$$

$$\mathbf{f}(t) = \begin{bmatrix} F_B(t) \\ F_D(t) \end{bmatrix} = \begin{bmatrix} F \sin(\omega t) \\ F \cos(\omega t) \end{bmatrix} = \text{Re}(\mathbf{F}e^{i\omega t}) \quad (6)$$

$$\mathbf{F} = \begin{bmatrix} -i \\ 1 \end{bmatrix} F \quad (7)$$

The last statement of (6) follows because, $\text{Re}(-iFe^{i\omega t}) = \text{Re}(-iF(\cos(\omega t) + i\sin(\omega t))) = F \sin(\omega t)$ and $\text{Re}(Fe^{i\omega t}) = \text{Re}(F(\cos(\omega t) + i\sin(\omega t))) = F \cos(\omega t)$.

Question 2:

The circular eigenfrequencies ω_j and the eigenmodes $\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix}$ are obtained as non-trivial solutions of the homogeneous linear equations, cf. (3-42)

$$\begin{bmatrix} 3 - \lambda_j & 1 \\ 1 & 2 - 2\lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (8)$$

$$\lambda_j = \frac{8 \omega_j^2 m a^3}{9 EI} \quad (9)$$

The frequency condition becomes

$$(3 - \lambda_j)(2 - 2\lambda_j) - 1 = 0 \Rightarrow \lambda_j = \begin{cases} \frac{1}{2}(4 - \sqrt{6}) & , \quad j = 1 \\ \frac{1}{2}(4 + \sqrt{6}) & , \quad j = 2 \end{cases} \quad (10)$$

$$\omega_j = \begin{cases} \frac{3}{4} \sqrt{4 - \sqrt{6}} \sqrt{\frac{EI}{ma^3}} & , \quad j = 1 \\ \frac{3}{4} \sqrt{4 + \sqrt{6}} \sqrt{\frac{EI}{ma^3}} & , \quad j = 2 \end{cases} \quad (11)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix} \quad (12)$$

The first component $\Phi_1^{(j)}$ is determined from the first equation of (8) as follows

$$(3 - \lambda_j)\Phi_1^{(j)} + 1 \cdot 1 = 0 \Rightarrow \Phi_1^{(j)} = -\frac{1}{3 - \lambda_j} = \begin{cases} -\sqrt{6} + 2 & , \quad j = 1 \\ \sqrt{6} + 2 & , \quad j = 2 \end{cases} \quad (13)$$

Question 3:

The stationary response of (3) becomes, see (3-100), (3-101)

$$\mathbf{x}(t) = \text{Re}(\mathbf{X}e^{i\omega t}) \quad (14)$$

$$\mathbf{X} = \mathbf{H}(\omega)\mathbf{F} \quad (15)$$

where F is given by (7). It follows from (4), (13) that $\Phi^{(1)T} \mathbf{C} \Phi^{(2)} = -2c \neq 0$. Hence, the expansion (3-196) of the frequency response matrix $\mathbf{H}(\omega)$ in terms of the undamped eigenmodes cannot be applied. Instead a straightforward evaluation of this quantity must be performed, cf. (3-102)

$$\mathbf{H}(\omega) = (\mathbf{K} - \omega^2 \mathbf{M} + i\omega \mathbf{C})^{-1} = \begin{bmatrix} \frac{27}{8} \frac{EI}{a^3} - m\omega^2 + i\omega c & \frac{9}{8} \frac{EI}{a^3} \\ \frac{9}{8} \frac{EI}{a^3} & \frac{18}{8} \frac{EI}{a^3} - 2m\omega^2 \end{bmatrix}^{-1} =$$

$$\frac{1}{D} \begin{bmatrix} \frac{18}{8} \frac{EI}{a^3} - 2m\omega^2 & -\frac{9}{8} \frac{EI}{a^3} \\ -\frac{9}{8} \frac{EI}{a^3} & \frac{27}{8} \frac{EI}{a^3} - m\omega^2 + i\omega c \end{bmatrix} \quad (16)$$

$$D = \left(\frac{27}{8} \frac{EI}{a^3} - m\omega^2 + i\omega c \right) \left(\frac{18}{8} \frac{EI}{a^3} - 2m\omega^2 \right) - \left(\frac{9}{8} \frac{EI}{a^3} \right)^2 \quad (17)$$

The amplitude vector (15) is written in the form

$$\mathbf{X} = \begin{bmatrix} |X_1| e^{-i\Psi_1} \\ |X_2| e^{-i\Psi_2} \end{bmatrix} \quad (18)$$

The displacement response (14) can then be written

$$\mathbf{x}(t) = \begin{bmatrix} \text{Re}(|X_1| e^{-i\Psi_1} e^{i\omega t}) \\ \text{Re}(|X_2| e^{-i\Psi_2} e^{i\omega t}) \end{bmatrix} = \begin{bmatrix} |X_1| \cos(\omega t - \Psi_1) \\ |X_2| \cos(\omega t - \Psi_2) \end{bmatrix} \quad (19)$$

The amplitudes $|X_i|$ and phases Ψ_i are determined, equating (15) and (18). In doing this, it should be noticed that the determinant D is complex.

PROBLEM 2

Question 1:

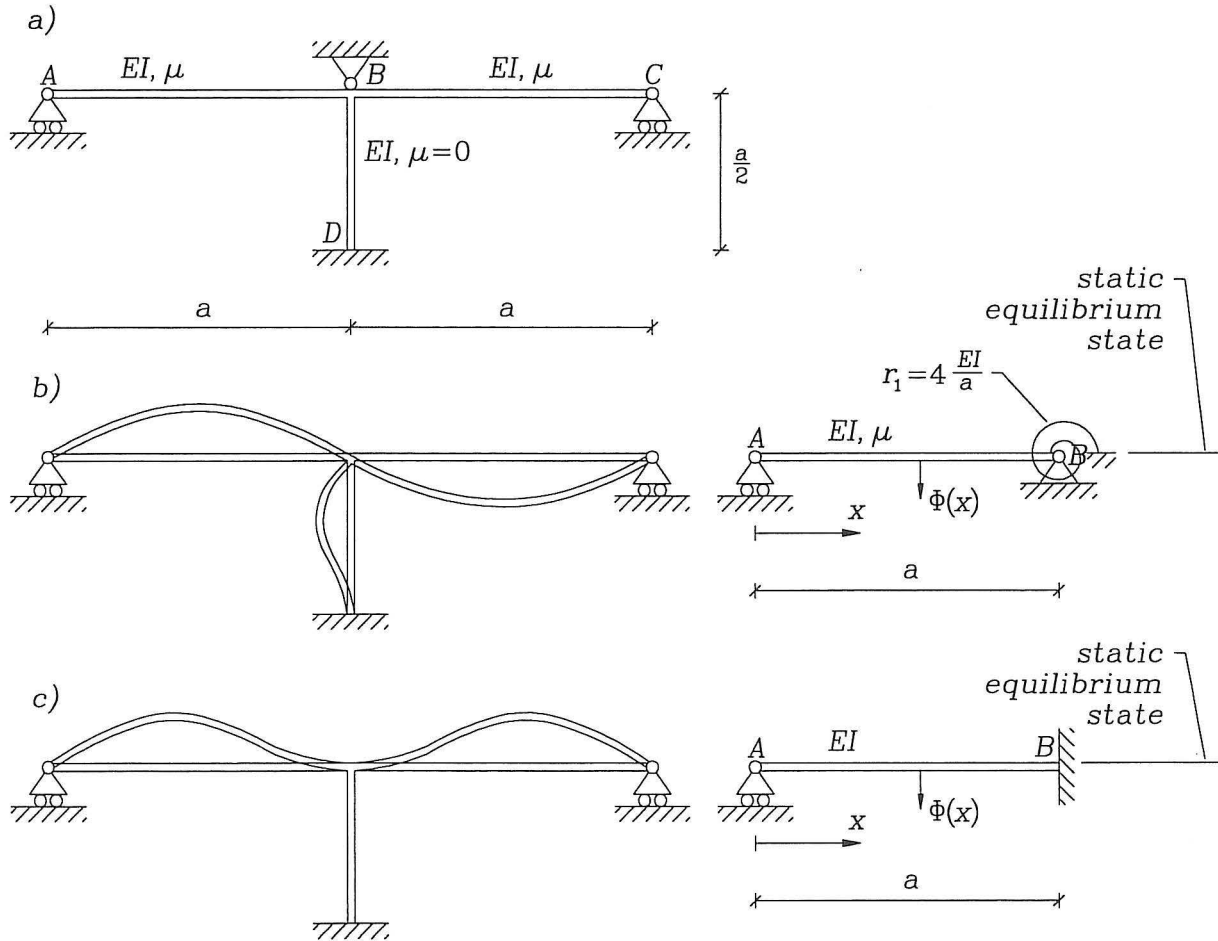


Fig. 1: a) Symmetric structural system. b) Anti-symmetric eigenvibrations and equivalent system. c) Symmetric eigenvibrations and equivalent system.

The system is mechanically and geometrically symmetric around the line BD . Hence, the eigenvibrations separate into anti-symmetric and symmetric eigenvibrations as shown in figs. 1b and 1c.

The equivalent systems for the analysis of anti-symmetric and symmetric eigenvibrations have also been indicated in figs. 1b and 1c. These equivalent beams have constant bending stiffness EI and constant mass per unit length μ . Hence, the eigenmodes for both equivalent systems are given by (4-18), (4-19)

$$\Phi(x) = A \sin\left(\lambda \frac{x}{a}\right) + B \cos\left(\lambda \frac{x}{a}\right) + C \sinh\left(\lambda \frac{x}{a}\right) + D \cosh\left(\lambda \frac{x}{a}\right) \quad (1)$$

$$\lambda^4 = \frac{\mu \omega^2 a^4}{EI} \quad (2)$$

The boundary conditions at point A ($x = 0$) for both equivalent systems read, see (4-23)

$$\Phi(0) = \frac{d^2}{dx^2}\Phi(0) = 0 \quad (3)$$

(3) implies that $B = D = 0$, see (4-24). Then (1) reduces to

$$\Phi(x) = A \sin\left(\lambda \frac{x}{a}\right) + C \sinh\left(\lambda \frac{x}{a}\right) \quad (4)$$

The anti-symmetric eigenvibrations, as shown in fig. 1b, have zero displacement at point B . Because the nodal point B has been fixed against horizontal displacements, the massless column BD is equivalent to a rotational spring with the spring constant $4.0 \frac{EI}{a/2} = 8.0 \frac{EI}{a}$. This spring stiffness is shared in equal parts by the two beam elements AB and BC during the anti-symmetric eigenvibrations. Hence, the spring stiffness $r_1 = 4.0 \frac{EI}{a}$ should be applied to the equivalent reduced system as shown in fig. 1b. The boundary conditions at point B ($x = a$) for the reduced equivalent system then become, see (4-13)

$$\Phi(a) = 0 \quad (5)$$

$$EI \frac{d^2}{dx^2}\Phi(a) = -r_1 \frac{d}{dx}\Phi(a) \quad (6)$$

Inserting (4) into (5) and (6) the following homogeneous equations are obtained for the determination of the constants A and C

$$\left. \begin{aligned} A \sin \lambda + C \sinh \lambda &= 0 \\ EI \frac{\lambda^2}{a^2} (-A \sin \lambda + C \sinh \lambda) &= -4 \frac{EI \lambda}{a} (A \cos \lambda + C \cosh \lambda) \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} \sin \lambda & \sinh \lambda \\ \lambda \sin \lambda - 4 \cos \lambda & -\lambda \sinh \lambda - 4 \cosh \lambda \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (7)$$

Non-trivial solutions of (7) are obtained if, and only if the determinant of the coefficient matrix becomes equal to zero. The frequency condition then reads

$$\sin \lambda (-\lambda \sinh \lambda - 4 \cosh \lambda) - \sinh \lambda (\lambda \sin \lambda - 4 \cos \lambda) = 0 \Rightarrow$$

$$\frac{\lambda}{2} \tan \lambda \tanh \lambda + \tan \lambda - \tanh \lambda = 0 \quad (8)$$

The 3 lowest solutions of (8), λ_1 , λ_3 , λ_5 , have been shown below in table 1, where the corresponding eigenmodes have also been sketched.

The symmetric eigenvibrations, as shown in fig. 1c, have zero displacement and zero slope at point B . Hence, the boundary conditions at point $B (x = a)$ for the reduced equivalent system then become, cf. (4-36)

$$\Phi(a) = 0 \quad (9)$$

$$\frac{d}{dx}\Phi(a) = 0 \quad (10)$$

Inserting (4) into (9) and (10) the following homogeneous equations are obtained for the determination of the constants A and C

$$\left. \begin{aligned} A \sin \lambda + C \sinh \lambda &= 0 \\ \frac{\lambda}{a} (A \cos \lambda + C \cosh \lambda) &= 0 \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} \sin \lambda & \sinh \lambda \\ \cos \lambda & \cosh \lambda \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (11)$$

The frequency condition then reads

$$\sin \lambda \cosh \lambda - \sinh \lambda \cos \lambda = 0 \Rightarrow$$

$$\tan \lambda - \tanh \lambda = 0 \quad (12)$$

The 3 lowest solutions of (12), $\lambda_2, \lambda_4, \lambda_6$, have been shown below in table 1, along with a sketch of the corresponding eigenmodes. As seen the anti-symmetric and symmetric eigenvibrations are changing.

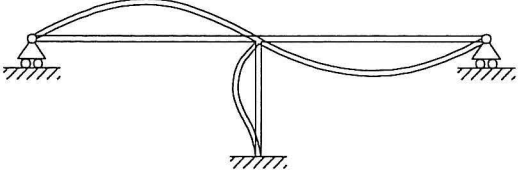
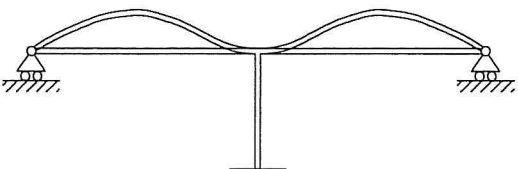
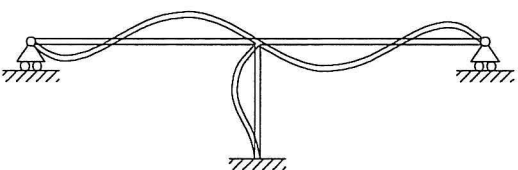
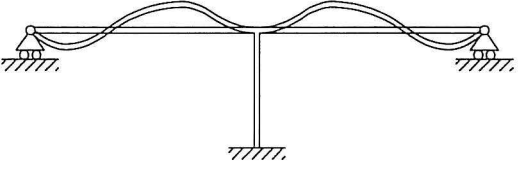
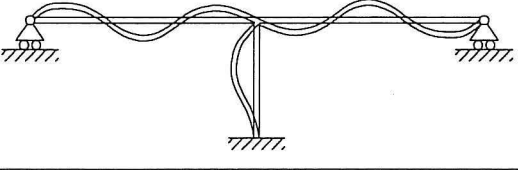
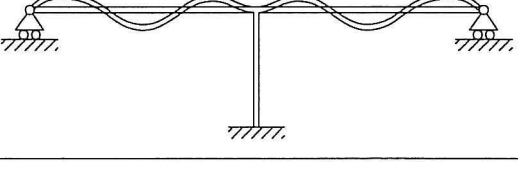
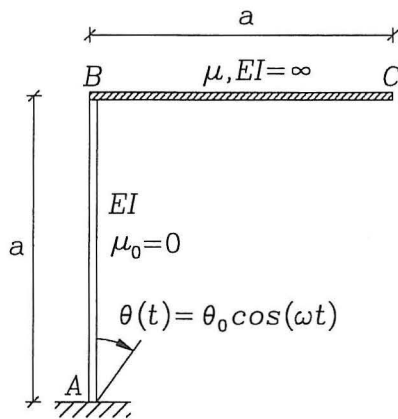
i	λ_i	<i>Eigenmode</i>
1	3.4907	
2	3.9266	
3	6.5139	
4	7.0686	
5	9.5956	
6	10.2102	

Table 1: Eigenvalues and eigenmodes of the system.

2.3 June 20, 1988

Duration: 3h

PROBLEM 1¹

The figure shows a plane beam system consisting of two sub-beams AB and BC both of the length a . The beam AB is a vertical massless linear elastic Bernoulli-Euler beam with the constant bending stiffness EI . The beam BC is a horizontal infinitely stiff beam with the constant mass per unit length μ . Both sub-beams are infinitely stiff against axial elongations. The structure is fixed at point A and free at point C . The structure is excited by a harmonic rotation $\theta(t) = \theta_0 \cos(\omega t)$ of the support point A with a sign as defined in the figure. Only small vibrations in the plane of the structure are considered. The influence on the bending moments from any normal forces in the beam AB , as well as the influence of any damping mechanism is ignored.

Question 1 (20%, $\mu = 3.5\%$)

Determine the equations of motion of the structure.

Question 2 (13%, $\mu = 5.4\%$)

Determine the undamped circular eigenfrequencies and the eigenmodes of the structure.

¹Students have coined the nickname the gallows for this problem may be due to the modest score in the first question.

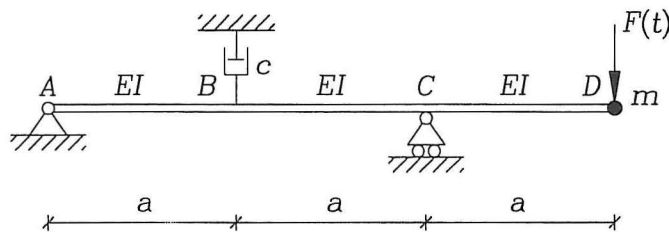
PROBLEM 2

Now, it is assumed that the beam AB is no longer massless, but has the constant mass per unit length $\mu_0 \neq 0$.

Question 1 (33%, $\mu = 19.5\%$)

Formulate the frequency condition for the determination of the undamped circular eigenfrequencies of the structure.

PROBLEM 3



The figure shows a plane horizontal massless linear elastic Bernoulli-Euler beam $ABCD$ of the length $3a$, and free of damping. The beam has constant bending stiffness EI , and is simply supported at one end at point A . At the third point B at the distance a from point A a linear viscous damper with the damping constant c is applied. At the other third point C at the distance a from point B a simple support is present. At the end of the cantilever part of the beam at point D a point mass m is applied. At point D the beam is subjected to the external vertical dynamic load $F(t)$. Only small vibrations from the static equilibrium state in the vertical direction are considered.

Question 1 (10%, $\mu = 7.8\%$)

Formulate the equations of motion for the vertical displacement of the points B and D from the static equilibrium state.

Question 2 (10%, $\mu = 5.7\%$)

Assume that the external load is harmonic with the amplitude F_0 and circular frequency ω

$$F(t) = F_0 \cos(\omega t)$$

Determine the stationary displacement response of point D , when any influence from the initial conditions has been dissipated.

SOLUTIONS

PROBLEM 1

Question 1:

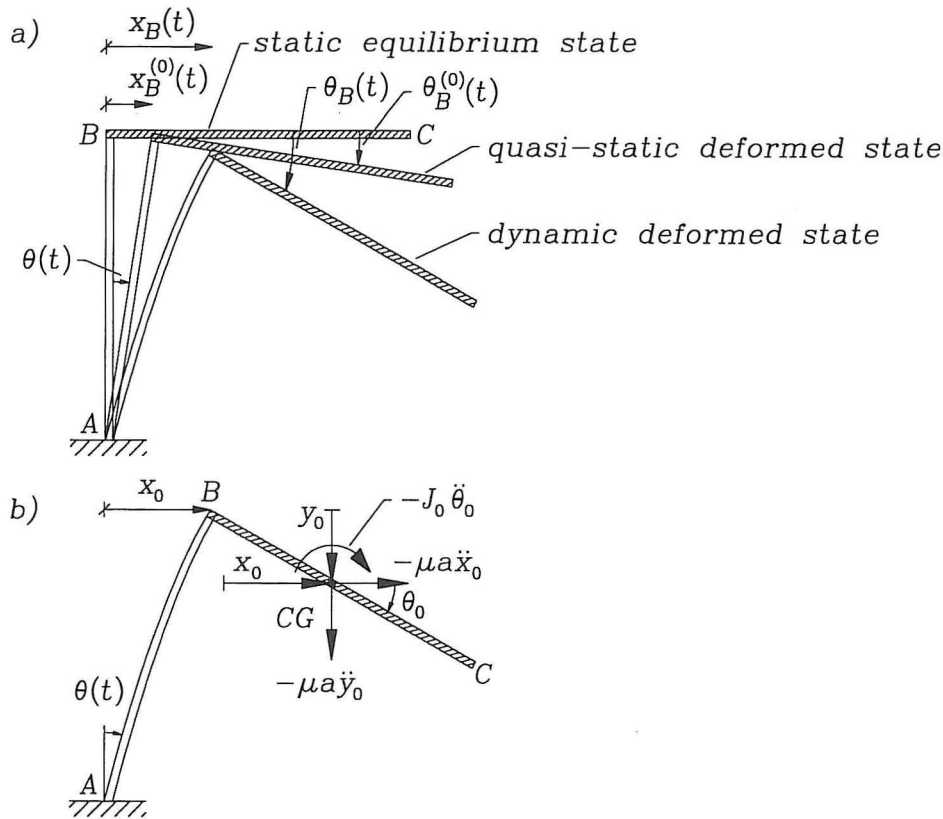


Fig. 1: a) Kinematic description of the system. b) Inertial loads on the system.

Since the beam AB is assumed to be massless in the present problem, the system has but 2 degrees of freedom which are selected as the horizontal displacement $x_B(t)$ and the rotation θ_B of point B from the static equilibrium state with signs as shown in figure 1a. These displacement components are made up of quasi-static components $x_B^{(0)}(t)$ and $\theta_B^{(0)}(t)$, caused by the stiff-body motion of the structure from the rotation $\theta(t)$ of point, and dynamic components from the inertial load on beam BC . The quasi-static motion can be written, see fig. 1a

$$\mathbf{x}^{(0)}(t) = \begin{bmatrix} x_B^{(0)}(t) \\ \theta_B^{(0)}(t) \end{bmatrix} = \begin{bmatrix} a \\ 1 \end{bmatrix} \theta(t) \quad (1)$$

The horizontal displacement $x_0(t)$, the vertical displacement $y_0(t)$ and the rotation

$\theta_0(t)$ of the centre of gravity CG of the beam BC from the static equilibrium state are introduced as auxiliary degrees of freedom with signs as defined in fig. 1b. These are given as

$$\left. \begin{aligned} x_0(t) &= x_B(t) \\ y_0(t) &= \theta_B(t) \frac{a}{2} \\ \theta_0(t) &= \theta_B(t) \end{aligned} \right\} \quad (2)$$

Next, the inertial forces $-\mu a\ddot{x}_0$ and $-\mu a\ddot{y}_0$, and the inertial moment $-J_0\ddot{\theta}_0$ with signs as shown in fig. 1b are applied as external loads acting in the centre of gravity CG according to d'Alembert's principle. $J_0 = \frac{1}{12}\mu a^3$ is the mass moment of inertia of the beam BC around CG . The equations of motion then read, cf. (3-328)

$$\left. \begin{aligned} x_B(t) &= x_B^{(0)}(t) + \delta_{x_B x_0}(-\mu a\ddot{x}_0) + \delta_{x_B y_0}(-\mu a\ddot{y}_0) + \delta_{x_B \theta_0}(-J_0\ddot{\theta}_0) \\ \theta_B(t) &= \theta_B^{(0)}(t) + \delta_{\theta_B x_0}(-\mu a\ddot{x}_0) + \delta_{\theta_B y_0}(-\mu a\ddot{y}_0) + \delta_{\theta_B \theta_0}(-J_0\ddot{\theta}_0) \end{aligned} \right\} \quad (3)$$

The flexibility coefficients $\delta_{x_B x_0}$, $\delta_{x_B y_0}$ and $\delta_{x_B \theta_0}$ indicate the displacement of point B from respectively a unit force at CG in the x_0 -direction, a unit force at CG in the y_0 -direction and a unit moment at CG in the θ_0 -direction. Correspondingly, $\delta_{\theta_B x_0}$, $\delta_{\theta_B y_0}$ and $\delta_{\theta_B \theta_0}$ indicate the rotation of point B from a unit force at CG in the x_0 -direction, a unit force at CG in the y_0 -direction and a unit moment at CG in the θ_0 -direction. From standard static analysis techniques these are found as

$$\left. \begin{aligned} \delta_{x_B x_0} &= \frac{a^3}{3EI} \quad , \quad \delta_{x_B y_0} = \frac{a^3}{4EI} \quad , \quad \delta_{x_B \theta_0} = \frac{a^2}{2EI} \\ \delta_{\theta_B x_0} &= \frac{a^2}{2EI} \quad , \quad \delta_{\theta_B y_0} = \frac{a^2}{2EI} \quad , \quad \delta_{\theta_B \theta_0} = \frac{a}{EI} \end{aligned} \right\} \quad (4)$$

Using (1), (2) and (4) the equations of motion can be written

$$\left. \begin{aligned} x_B(t) &= a\theta(t) - \frac{a^3}{EI} \left(\frac{1}{3}\mu a\ddot{x}_B + \frac{1}{6}\mu a^2\ddot{\theta}_B \right) \\ \theta_B(t) &= \theta(t) - \frac{a^2}{EI} \left(\frac{1}{2}\mu a\ddot{x}_B + \frac{1}{3}\mu a^2\ddot{\theta}_B \right) \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} 2 & 1 \\ 1 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \ddot{x}_B \\ a\ddot{\theta}_B \end{bmatrix} + \frac{EI}{\mu a^4} \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_B \\ a\theta_B \end{bmatrix} = \frac{EI}{\mu a^3} \begin{bmatrix} 6 \\ 2 \end{bmatrix} \theta(t) \quad (5)$$

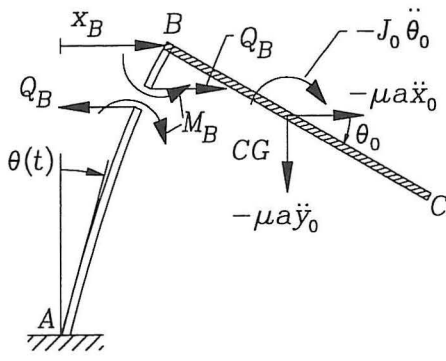


Fig. 2: Alternative approach. Loads on the free beam BC .

The beam BC is cut free, and the shear force Q_B and the bending moment M_B in the beam AB at point B are applied as external loads with signs as shown in fig. 2. Alternatively, the equations of motion can then be derived, expressing that Q_B and M_B must be in equilibrium with the previously mentioned inertial loads acting at the centre of gravity CG of the beam BC , see fig. 2

$$\left. \begin{aligned} Q_B - \mu a \ddot{x}_0 &= 0 \\ M_B + J_0 \ddot{\theta}_0 + \mu a \ddot{y}_0 \cdot \frac{a}{2} &= 0 \end{aligned} \right\} \Rightarrow \left. \begin{aligned} Q_B - \mu a \ddot{x}_B &= 0 \\ M_B + \frac{1}{3} \mu a^3 \ddot{\theta}_B &= 0 \end{aligned} \right\} \quad (6)$$

where (2) has been applied. Now, Q_B and M_B are caused by the dynamic displacement $(x_B - a\theta)$ and dynamic rotation $(\theta_B - \theta)$ of node B , since the stiff-body motion is not introducing any stresses in the structure. The following well-known stiffness relations then apply

$$\left. \begin{aligned} Q_B &= -12 \frac{EI}{a^3} (x_B - a\theta) + 6 \frac{EI}{a^2} (\theta_B - \theta) \\ M_B &= -6 \frac{EI}{a^2} (x_B - a\theta) + 4 \frac{EI}{a} (\theta_B - \theta) \end{aligned} \right\} \quad (7)$$

Inserting (7) into (6) the following equations of motion are obtained

$$\begin{bmatrix} \ddot{x}_B \\ a\ddot{\theta}_B \end{bmatrix} + 6 \frac{EI}{\mu a^4} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x_B \\ a\theta_B \end{bmatrix} = 6 \frac{EI}{\mu a^3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \theta(t) \quad (8)$$

(8) can be reduced to (5) by pre-multiplication by the matrix $\begin{bmatrix} 2 & 1 \\ 1 & \frac{2}{3} \end{bmatrix}$.

Question 2:

Eigenvibrations are given as, cf. (3-54)

$$\begin{bmatrix} x_B(t) \\ a\theta_B(t) \end{bmatrix} = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \cos(\omega t) \quad (9)$$

where the eigenmodes are the solution to the linear homogeneous equations, see eq. (8)

$$\begin{bmatrix} 2 - \lambda_j & -1 \\ -3 & 2 - \lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (10)$$

$$\lambda_j = \frac{1}{6} \frac{\mu a^4 \omega_j^2}{EI} \quad (11)$$

The characteristic equation becomes

$$(2 - \lambda_j)^2 - 3 \cdot 1 = 0 \Rightarrow$$

$$\lambda_j = \begin{cases} 2 - \sqrt{3} & , \quad j = 1 \\ 2 + \sqrt{3} & , \quad j = 2 \end{cases} \quad (12)$$

$$\omega_j = \begin{cases} \sqrt{6(2 - \sqrt{3})} \sqrt{\frac{EI}{\mu a^4}} & , \quad j = 1 \\ \sqrt{6(2 + \sqrt{3})} \sqrt{\frac{EI}{\mu a^4}} & , \quad j = 2 \end{cases} \quad (13)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix} \quad , \quad j = 1, 2 \quad (14)$$

The first component $\Phi_1^{(j)}$ is determined from the first equation (10)

$$(2 - \lambda_j) \Phi_1^{(j)} - 1 \cdot 1 = 0 \Rightarrow$$

$$\Phi_1^{(j)} = \frac{1}{2 - \lambda_j} = \begin{cases} \frac{\sqrt{3}}{3} & , \quad j = 1 \\ -\frac{\sqrt{3}}{3} & , \quad j = 2 \end{cases} \quad (15)$$

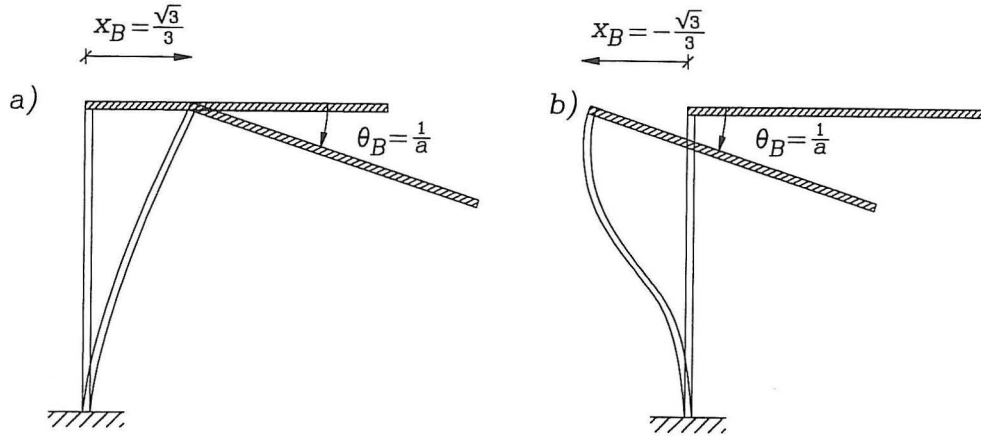


Fig. 3: a) 1st eigenmode. b) 2nd eigenmode.

PROBLEM 2

Question 1:

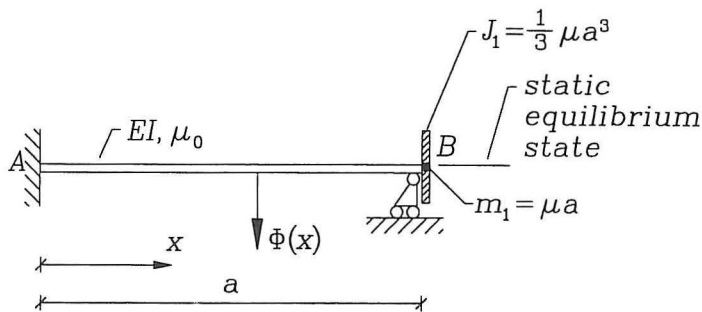


Figure 1: Equivalent system.

The effect of the infinitely stiff beam BC is equivalent to a distributed mass attached to point B with the mass $m_1 = \mu a$ and the mass moment of inertia $J_1 = \frac{1}{3} \mu a^3$ around point B . Sections on the beam AB are identified by the x -axis orientated from point A as shown in figure 1.

Because the beam AB has constant bending stiffness EI , constant mass per unit length μ_0 and the normal force N can be ignored, the eigenmodes are given as, cf. (4-18), (4-19)

$$\Phi(x) = A \sin\left(\lambda \frac{x}{a}\right) + B \cos\left(\lambda \frac{x}{a}\right) + C \sinh\left(\lambda \frac{x}{a}\right) + D \cosh\left(\lambda \frac{x}{a}\right) \quad (1)$$

$$\lambda^4 = \frac{\mu_0 \omega^2 a^4}{EI} \quad (2)$$

The geometrical boundary conditions at point $A(x = 0)$ become cf. (4-36)

$$\Phi(0) = \frac{d}{dx}\Phi(0) = 0 \quad (3)$$

Inserting (1) into (3) implies that $C = -A$ and $D = -B$. Then (1) reduces to

$$\Phi(x) = A \left(\sin\left(\lambda \frac{x}{a}\right) - \sinh\left(\lambda \frac{x}{a}\right) \right) + B \left(\cos\left(\lambda \frac{x}{a}\right) - \cosh\left(\lambda \frac{x}{a}\right) \right) \quad (4)$$

The mechanical boundary conditions at point $B(x = a)$ become, see (4-13)

$$\left. \begin{aligned} EI \frac{d^2}{dx^2} \Phi(a) &= \omega^2 J_1 \frac{d}{dx} \Phi(a) \quad , \quad J_1 = \frac{1}{3} \mu a^2 \\ EI \frac{d^3}{dx^3} \Phi(a) &= -\omega^2 m_1 \Phi(a) \quad , \quad m_1 = \mu a \end{aligned} \right\} \quad (5)$$

By inserting (4) into (5) and eliminating ω^2 in the favour of the frequency parameter λ by means of (2), the following homogeneous equations are then obtained for the determination of the coefficients A and B

$$\left. \begin{aligned} EI \frac{\lambda^2}{a^2} \left(A(-\sin \lambda - \sinh \lambda) + B(-\cos \lambda - \cosh \lambda) \right) &= \\ \omega^2 \cdot \frac{1}{3} \mu a^3 \cdot \frac{\lambda}{a} \left(A(\cos \lambda - \cosh \lambda) + B(-\sin \lambda - \sinh \lambda) \right) &= \\ EI \frac{\lambda^3}{a^3} \left(A(-\cos \lambda - \cosh \lambda) + B(\sin \lambda - \sinh \lambda) \right) &= \\ -\omega^2 \cdot \mu a \cdot \frac{\lambda}{a} \left(A(\sin \lambda - \sinh \lambda) + B(\cos \lambda - \cosh \lambda) \right) &= \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} K_{11}(\lambda) & K_{12}(\lambda) \\ K_{21}(\lambda) & K_{22}(\lambda) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6)$$

$$K_{11}(\lambda) = \sin \lambda + \sinh \lambda + \frac{1}{3} \frac{\mu}{\mu_0} \lambda^3 (\cos \lambda - \cosh \lambda) \quad (6a)$$

$$K_{12}(\lambda) = \cos \lambda + \cosh \lambda - \frac{1}{3} \frac{\mu}{\mu_0} \lambda^3 (\sin \lambda + \sinh \lambda) \quad (6b)$$

$$K_{21}(\lambda) = \cos \lambda + \cosh \lambda - \frac{\mu}{\mu_0} \lambda (\sin \lambda - \sinh \lambda) \quad (6c)$$

$$K_{22}(\lambda) = -\sin \lambda + \sinh \lambda - \frac{\mu}{\mu_0} \lambda (\cos \lambda - \cosh \lambda) \quad (6d)$$

Non-trivial solutions $A \neq 0 \vee B \neq 0$ of (6) are obtained, if the determinant of the coefficient matrix is zero. The frequency condition then becomes

$$K_{11}(\lambda)K_{22}(\lambda) - K_{12}(\lambda)K_{21}(\lambda) = 0 \quad (7)$$

Using well-known trigonometric and hyperbolic identities, (7) can be reduced to the following form

$$\begin{aligned} 1 + \frac{1}{3}\mu^2\lambda^4 + \cos\lambda \cosh\lambda \left(1 - \frac{1}{3}\mu^2\lambda^4\right) - \sin\lambda \sinh\lambda \left(1 + \frac{1}{3}\lambda^2\right) \frac{\mu}{\mu_0}\lambda + \\ \cos\lambda \sinh\lambda \left(1 - \frac{1}{3}\lambda^2\right) \frac{\mu}{\mu_0}\lambda = 0 \end{aligned} \quad (8)$$

For $\mu_0 = \mu$, the following solutions of (8) can be obtained by means of a suitable iteration scheme

$$\lambda_j = \begin{cases} 1.09667 & , \quad j = 1 \\ 2.05031 & , \quad j = 2 \\ 4.91718 & , \quad j = 3 \\ 7.97027 & , \quad j = 4 \\ 11.08039 & , \quad j = 5 \end{cases} \quad (9)$$

PROBLEM 3

Question 1:

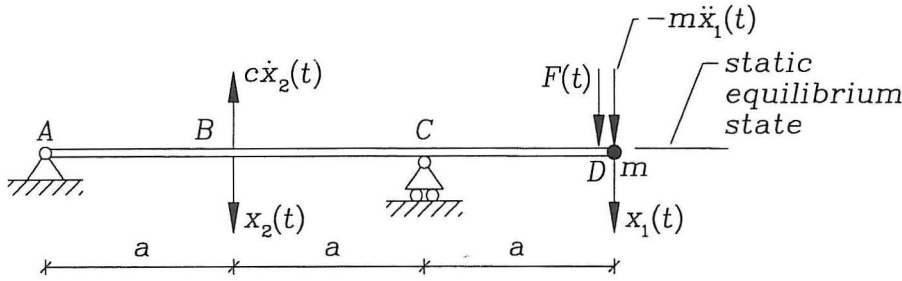


Fig. 1: Forces on a free beam.

The beam is massless. Hence, the system has but a single degree of freedom, which is selected as the vertical displacement $x_1(t)$ of point D from the static equilibrium state with a sign as defined in fig. 1. Besides, an auxiliary degree of freedom $x_2(t)$ is introduced for the vertical displacement of point B from the static equilibrium state. The beam is cut free from the damper, and the damper force $c\dot{x}_2(t)$ is applied as an external force with a sign as defined in fig. 1. Further, the inertial load $-m\ddot{x}_1(t)$ is

applied as an external load in accordance with d'Alembert's principle. The equations of motion then read, cf. (3-1)

$$\left. \begin{aligned} x_1(t) &= \delta_{11}(F(t) - m\ddot{x}_1) + \delta_{12}(-c\dot{x}_2) \\ x_2(t) &= \delta_{21}(F(t) - m\ddot{x}_1) + \delta_{22}(-c\dot{x}_2) \end{aligned} \right\} \quad (1)$$

The flexibility coefficients are given as, see (B-1), (B-2), (B-3)

$$\mathbf{D} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \frac{1}{12} \frac{a^3}{EI} \begin{bmatrix} 12 & -3 \\ -3 & 2 \end{bmatrix} \quad (2)$$

The equations of motion for the displacements of point B and point D can then be written

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \quad (3)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \quad (4)$$

$$\mathbf{f}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F(t), \quad \mathbf{K} = \frac{4}{5} \frac{EI}{a^3} \begin{bmatrix} 2 & 3 \\ 3 & 12 \end{bmatrix} \quad (5)$$

Jocularly, the present system has been described as one having 1.5 degrees of freedom.

Question 2:

With $F(t) = F_0 \cos(\omega t)$, the load vector can be written

$$\mathbf{f}(t) = \mathbf{F}_0 \cos(\omega t) \quad (6)$$

$$\mathbf{F}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_0 \quad (7)$$

The stationary response of (3) becomes, see (3-100), (3-101)

$$\mathbf{x}(t) = \text{Re}(\mathbf{X}_0 e^{i\omega t}) \quad (8)$$

$$\mathbf{X}_0 = \mathbf{H}(\omega) \mathbf{F}_0 \quad (9)$$

$$\mathbf{H}(\omega) = (\mathbf{K} - \omega^2 \mathbf{M} + i\omega \mathbf{C})^{-1} = \frac{1}{D} \begin{bmatrix} \frac{48}{5} \frac{EI}{a^3} + i\omega c & -\frac{12}{5} \frac{EI}{a^3} \\ -\frac{12}{5} \frac{EI}{a^3} & \frac{8}{5} \frac{EI}{a^3} + (i\omega)^2 m \end{bmatrix} \quad (10)$$

$$D = \left(\frac{8}{5} \frac{EI}{a^3} + (i\omega)^2 m \right) \left(\frac{48}{5} \frac{EI}{a^3} + i\omega c \right) - \left(\frac{12}{5} \frac{EI}{a^3} \right)^2 \quad (11)$$

From (6), (7), (8), (9) it then follows that the stationary response of the point D becomes

$$x_1(t) = \text{Re}(H_{11}(\omega) F_0 e^{i\omega t}) \quad (12)$$

$$H_{11}(\omega) = \frac{P(i\omega)}{Q(i\omega)} \quad (13)$$

$$P(z) = p_0 z + p_1 \quad (14)$$

$$Q(z) = z^3 + q_1 z^2 + q_2 z + q_3 \quad (15)$$

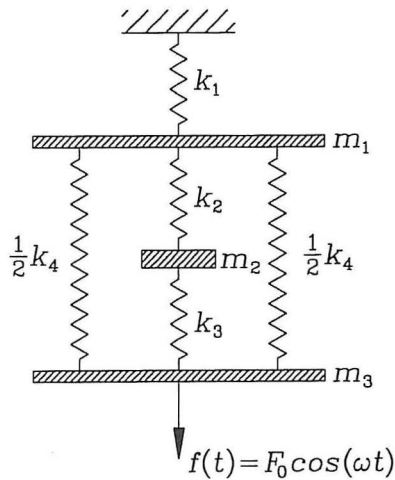
$$p_0 = \frac{1}{m} \quad , \quad p_1 = \frac{48}{5} \frac{EI}{mca^3} \quad (16)$$

$$q_1 = \frac{48}{5} \frac{EI}{ca^3} \quad , \quad q_2 = \frac{8}{5} \frac{EI}{ma^3} \quad , \quad q_3 = \frac{48}{5} \frac{1}{mc} \left(\frac{EI}{a^3} \right)^2 \quad (17)$$

2.4 September 14, 1988

Duration: 3h

PROBLEM 1



The system shown in the figure consists of 3 concentrated masses m_1, m_2 and m_3 , suspended vertically below each other. m_1 is connected to a linear elastic spring with the spring constant k_1 , which is fixed at the other end. m_1 and m_2 are connected with a linear elastic spring with the spring constant k_2 . m_2 and m_3 are connected with a linear elastic spring with the spring constant k_3 . Finally, m_1 and m_3 are connected by 2 linear elastic springs, each with the spring constant $\frac{1}{2}k_4$. It is assumed that the masses only move in the vertical direction.

Question 1 (20%, $\mu = 13.9\%$)

The mass m_3 is excited by an external vertical harmonic varying dynamic load $f(t) = F_0 \cos(\omega t)$ with the amplitude F_0 and the circular frequency ω . It is assumed that the load has been acting for such period of time that the response from the initial conditions has been dissipated. Otherwise, the effect of any damping mechanism is ignored. Analyse the possibilities for choosing k_4 , so that the mass m_1 has the motion 0, and determine the motion of the remaining masses.

Question 2 (13%, $\mu = 7.6\%$)

Determine the undamped circular eigenfrequencies and the eigenmodes of the system for the parameter values

$$m_1 = m_2 = m_3 = m$$

$$k_1 = k_2 = k_3 = k$$

SOLUTIONS

PROBLEM 1

Question 1:

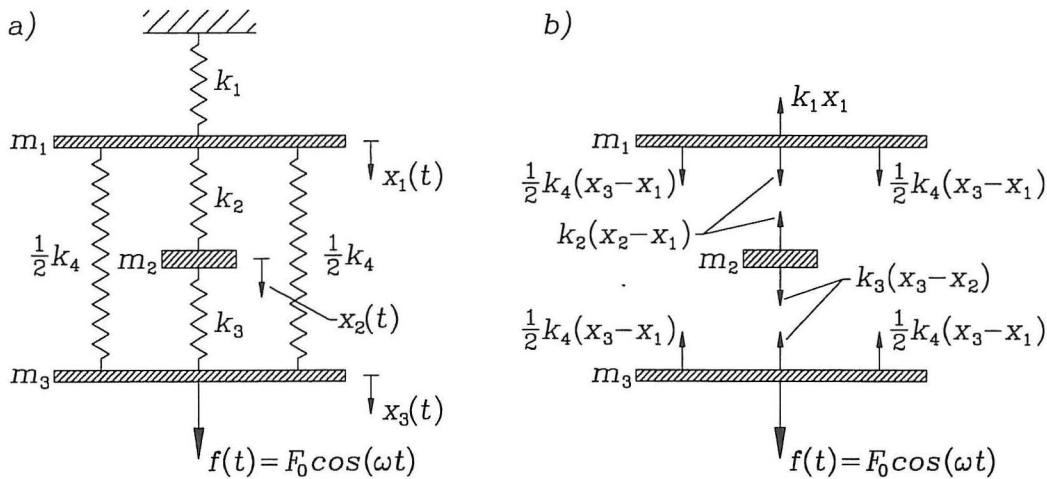


Fig. 1: a) Definition of degrees of freedom. b) Forces on the free masses.

The system has 3 degrees of freedom, which are selected as the vertical displacements $x_1(t)$, $x_2(t)$ and $x_3(t)$ of the masses from the static equilibrium state with signs as defined in fig. 1a. The masses are cut free from the springs and the spring forces due to the relative motion of the masses are applied as external forces. The magnitude and the sign of the spring forces have been defined in fig. 1b. The equations of motion are then obtained applying Newton's 2nd law of motion to each of the 3 masses

$$\left. \begin{aligned} m_1 \ddot{x}_1 &= 2 \cdot \frac{1}{2} k_4 (x_3 - x_1) + k_2 (x_2 - x_1) - k_1 x_1 \\ m_2 \ddot{x}_2 &= k_3 (x_3 - x_2) - k_2 (x_2 - x_1) \\ m_3 \ddot{x}_3 &= -2 \cdot \frac{1}{2} k_4 (x_3 - x_1) - k_3 (x_3 - x_2) + f(t) \end{aligned} \right\} \Rightarrow$$

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) = \mathbf{F}_0 \cos(\omega t) \quad (1)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad \mathbf{F}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} F_0 \quad (2)$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 + k_4 & -k_2 & -k_4 \\ -k_2 & k_2 + k_3 & -k_3 \\ -k_4 & -k_3 & k_3 + k_4 \end{bmatrix} \quad (3)$$

With $\mathbf{f}(t) = \mathbf{F}_0 \cos(\omega t)$ the stationary displacement response of the masses is given as, cf. (3-100), (3-101)

$$\mathbf{x}(t) = \mathbf{X} \cos(\omega t) \quad (4)$$

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \mathbf{H}(\omega) \mathbf{F}_0 \quad (5)$$

$$\mathbf{H}(\omega) = (\mathbf{K} - \omega^2 \mathbf{M})^{-1} = \frac{1}{D} \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{bmatrix} \quad (6)$$

$$D = (k_1 + k_2 + k_3 - \omega^2 m_1) \left((k_2 + k_3 - \omega^2 m_2)(k_3 + k_4 - \omega^2 m_3) - k_3^2 \right) - k_4^2 (k_2 + k_3 - \omega^2 m_2) - k_2^2 (k_3 + k_4 - \omega^2 m_3) - 2k_2 k_3 k_4 \quad (6a)$$

$$h_{11} = (k_2 + k_3 - \omega^2 m_2)(k_3 + k_4 - \omega^2 m_3) - k_3^2 \quad (6b)$$

$$h_{12} = k_2(k_3 + k_4 - \omega^2 m_3) + k_3 k_4 \quad (6c)$$

$$h_{13} = k_4(k_2 + k_3 - \omega^2 m_2) + k_2 k_3 \quad (6d)$$

$$h_{22} = (k_1 + k_2 + k_4 - \omega^2 m_1)(k_3 + k_4 - \omega^2 m_3) - k_4^2 \quad (6e)$$

$$h_{23} = k_3(k_1 + k_2 + k_4 - \omega^2 m_1) + k_2 k_4 \quad (6f)$$

$$h_{33} = (k_1 + k_2 + k_4 - \omega^2 m_1)(k_2 + k_3 - \omega^2 m_2) - k_2^2 \quad (6g)$$

$\mathbf{x}(t) = \mathbf{X} \cos(\omega t)$ is in phase with the excitation $\mathbf{f}(t) = \mathbf{F}_0 \cos(\omega t)$, and \mathbf{X} is real, because the structural system is free of damping. From (2), (5), (6) it follows that

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \frac{1}{D} \begin{bmatrix} h_{13} \\ h_{23} \\ h_{33} \end{bmatrix} F_0 \quad (7)$$

where D , h_{13} , h_{23} and h_{33} are given by (6a), (6d), (6f) and (6g).

$x_1(t) \equiv 0$, if and only if $X_1 = 0$. This is the case if $h_{13}(\omega) = 0$. From (6d) it follows that

$$h_{13}(\omega) = 0 \Leftrightarrow k_4(k_2 + k_3 - \omega^2 m_2) + k_2 k_3 = 0 \Leftrightarrow$$

$$k_4 = \frac{k_2 k_3}{\omega^2 m_2 - k_2 - k_3} \quad (8)$$

k_4 must be positive for any physically realizable solution. k_4 can then only be chosen so that the mass m_1 is at rest, if the denominator of (8) is positive, i.e. if the circular frequency of the excitation fulfils

$$\omega^2 > \frac{k_2 + k_3}{m_2} \quad (9)$$

At low frequency excitations, not fulfilling (9), m_1 will be moved for any choice of $k_4 \in]0, \infty[$. This is especially the case in the quasi-static case $\omega = 0$. If k_4 fulfils (8), one has

$$k_2 + k_3 - \omega^2 m_2 = -\frac{k_2 k_3}{k_4} \quad (10)$$

By inserting (10) into (6a), (6f), (6g) the following amplitudes of the displacement response are obtained after a little algebra

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \frac{1}{k_3 k_4 + k_2(k_3 + k_4 - \omega^2 m_3)} \begin{bmatrix} 0 \\ -k_4 \\ k_2 \end{bmatrix} \quad (11)$$

Question 2:

For $m_1 = m_2 = m_3 = m$ and $k_1 = k_2 = k_3 = k$ the circular eigenfrequencies and the eigenmodes are obtained as non-trivial solutions of the homogeneous linear equations, cf. (3-42), (2), (3)

$$\begin{bmatrix} 3 - \lambda_j & -1 & -1 \\ -1 & 2 - \lambda_j & -1 \\ -1 & -1 & 2 - \lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \\ \Phi_3^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (12)$$

$$\lambda_j = \omega_j^2 \frac{m}{k} \quad (13)$$

The frequency condition, expressing that the determinant of the coefficient matrix of (12) is equal to zero becomes

$$(3 - \lambda_j)(\lambda_j^2 - 4\lambda_j + 1) = 0 \Rightarrow$$

$$\lambda_j = \begin{cases} 2 - \sqrt{3} & , \quad j = 1 \\ 3 & , \quad j = 2 \\ 2 + \sqrt{3} & , \quad j = 3 \end{cases} \quad (14)$$

$$\omega_j = \begin{cases} \sqrt{2 - \sqrt{3}} \sqrt{\frac{k}{m}} & , \quad j = 1 \\ \sqrt{3} \sqrt{\frac{k}{m}} & , \quad j = 2 \\ \sqrt{2 + \sqrt{3}} \sqrt{\frac{k}{m}} & , \quad j = 3 \end{cases} \quad (15)$$

The eigennodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \\ 1 \end{bmatrix} \quad , \quad j = 1, 2, 3 \quad (16)$$

$\Phi_1^{(j)}$ and $\Phi_2^{(j)}$ are determined from the first 2 equations of (12)

$$\left. \begin{aligned} (3 - \lambda_j) \Phi_1^{(j)} - \Phi_1^{(j)} - 1 \cdot 1 &= 0 \\ -\Phi_1^{(j)} + (2 - \lambda_j) \Phi_2^{(j)} - 1 \cdot 1 &= 0 \end{aligned} \right\} \Rightarrow$$

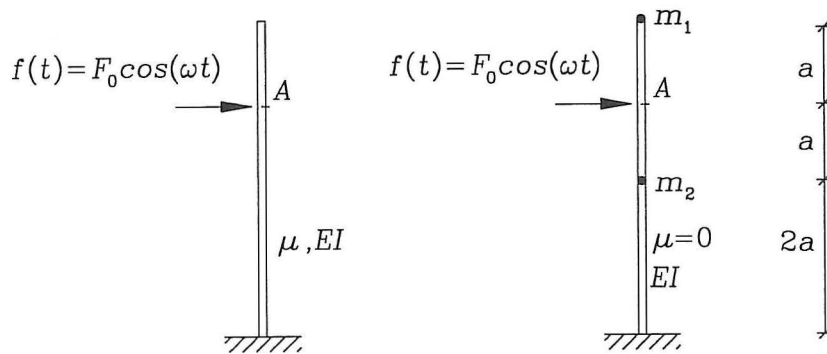
$$\Phi_1^{(j)} = \begin{cases} \sqrt{3} - 1 & , \quad j = 1 \\ 0 & , \quad j = 2 \\ -\sqrt{3} - 1 & , \quad j = 3 \end{cases} \quad (17)$$

$$\Phi_2^{(j)} = \begin{cases} 1 \\ -1 \\ 1 \end{cases} \quad (18)$$

2.5 June 16, 1989

Duration: 3h

PROBLEM 1



The figure to the left shows a plane vertical Bernoulli-Euler cantilever beam of the length $4a$. The beam has constant bending stiffness EI and constant mass per unit length μ . Vibrations in the axial direction and vibrations out of the plane are ignored.

The figure to the right shows a similar system with the only difference that the beam is assumed to be massless. Instead a concentrated mass m_1 is applied to the beam at the free end and a concentrated mass m_2 is applied at the mid-point.

Question 1 (25%)

Show that the 2 lowest undamped circular eigenfrequencies of the 2 systems become identical, if the concentrated masses are chosen as follows

$$m_1 = 0.869229\mu a \quad , \quad m_2 = 1.009398\mu a$$

Question 2 (13%)

The system to the right in the figure is excited by a horizontal harmonic varying force $f(t) = F_0 \cos(\omega t)$ with the amplitude F_0 and the circular frequency ω . The force is acting at the quarter point A at the distance a from the free end.

Determine the stationary displacement response of the point masses m_1 and m_2 , when these are given by the figures indicated in problem 1. It is assumed that the load has been acting for such period of time that the response from the initial conditions has been dissipated. Otherwise, the effect of any damping mechanism is ignored.

Question 3 (25%)

Next, the system to the left is also excited by the same harmonic varying load acting at the quarter point A .

Determine the stationary displacement response of the free end of the beam using modal analyses, where 2 modal coordinates are retained in the truncated modal expansion. Again, damping effects are ignored.

The following definite integral may be useful at the solution of the problem

$$\int_0^1 \left((\cos \lambda + \cosh \lambda) (\sin(\lambda x) - \sinh(\lambda x)) - (\sin \lambda + \sinh \lambda) (\cos(\lambda x) - \cosh(\lambda x)) \right)^2 dx =$$

$$\begin{cases} 17.1242 & , \quad \lambda = 1.8751 \\ 2877.82 & , \quad \lambda = 4.6941 \end{cases}$$

SOLUTIONS

PROBLEM 1

Question 1:

The undamped circular eigenfrequencies of a cantilevered homogeneous beam are given as, see (4-41), (4-42)

$$\lambda_j = \begin{cases} 1.87510 & , \quad j = 1 \\ 4.69409 & , \quad j = 2 \end{cases} \quad (1)$$

$$\omega_j^2 = \frac{\lambda_j^4}{256} \frac{EI}{a^3} \frac{1}{\mu a} = \begin{cases} 0.0482905 \frac{EI}{a^3} \frac{1}{\mu a} & , \quad j = 1 \\ 1.8965578 \frac{EI}{a^3} \frac{1}{\mu a} & , \quad j = 2 \end{cases} \quad (2)$$

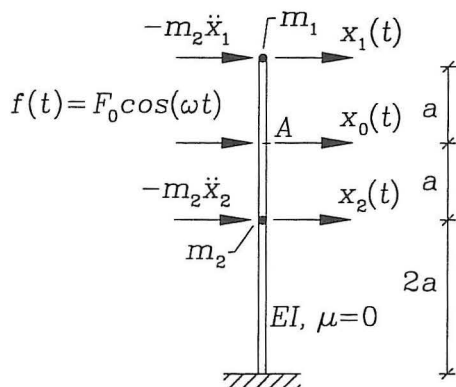


Fig. 1: Forces on the oscillating beam.

Since the beam is assumed to be massless, the system has but 2 degrees of freedom, which are selected as the horizontal displacements $x_1(t)$ and $x_2(t)$ of the masses from the static equilibrium state with signs as defined in fig. 1. Besides, an artificial degree of freedom $x_0(t)$ is introduced, indicating the horizontal displacement of point A from the static equilibrium state. Further, inertial loads $-m_1\ddot{x}_1$ and $-m_2\ddot{x}_2$ are applied as external forces at the masses with signs as shown in fig. 1, according to d'Alembert's principle. The equations of motion then read, cf. (3-343)

$$\left. \begin{aligned} x_1(t) &= \delta_{10}f(t) + \delta_{11}(-m_1\ddot{x}_1) + \delta_{12}(-m_2\ddot{x}_2) \\ x_2(t) &= \delta_{20}f(t) + \delta_{21}(-m_1\ddot{x}_1) + \delta_{22}(-m_2\ddot{x}_2) \end{aligned} \right\} \quad (3)$$

The flexibility coefficients are given as, cf. (B-5)

$$\mathbf{d}_0 = \begin{bmatrix} \delta_{10} \\ \delta_{20} \end{bmatrix} = \frac{1}{6} \frac{a^3}{EI} \begin{bmatrix} 81 \\ 28 \end{bmatrix} \quad (4)$$

$$\mathbf{D} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \frac{4}{3} \frac{a^3}{EI} \begin{bmatrix} 16 & 5 \\ 5 & 2 \end{bmatrix} \quad (5)$$

The equations of motion can then be written, cf. (3-349)

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{K}\mathbf{d}_0 F_0 \cos(\omega t) = \mathbf{F}_0 \cos(\omega t) \quad (6)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{K} = \mathbf{D}^{-1} = \frac{3}{28} \frac{EI}{a^3} \begin{bmatrix} 2 & -5 \\ -5 & 16 \end{bmatrix} \quad (7)$$

$$\mathbf{F}_0 = \mathbf{K}\mathbf{d}_0 F_0 = \frac{1}{56} \begin{bmatrix} 22 \\ 43 \end{bmatrix} F_0 \quad (8)$$

Undamped circular eigenfrequencies ω_j and eigenmodes $\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix}$ become, cf. (3-42)

$$\begin{bmatrix} 2 - \alpha_j m_1 & -5 \\ -5 & 16 - \alpha_j m_2 \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (9)$$

$$\alpha_j = \frac{28}{3} \frac{a^3}{EI} \omega_j^2 \quad (10)$$

The characteristic equation then becomes

$$(2 - \alpha_j m_1)(16 - \alpha_j m_2) - 25 = 0 \Rightarrow \left. \begin{aligned} \alpha_1 \\ \alpha_2 \end{aligned} \right\} = \frac{8m_1 + m_2 \mp \sqrt{64m_1^2 + 9m_1m_2 + m_2^2}}{m_1m_2} \quad (11)$$

Equating ω_1^2 and ω_2^2 as given by (10), (11) with ω_1^2 and ω_2^2 given by (2), the following 2 equations are obtained for m_1 and m_2

$$\left. \begin{aligned} 0.4507112 \frac{1}{\mu a} &= \frac{8m_1 + m_2 - \sqrt{64m_1^2 + 9m_1m_2 + m_2^2}}{m_1m_2} \\ 17.701206 \frac{1}{\mu a} &= \frac{8m_1 + m_2 + \sqrt{64m_1^2 + 9m_1m_2 + m_2^2}}{m_1m_2} \end{aligned} \right\} \quad (12)$$

Adding and subtracting these equations the following equivalent equations are obtained

$$9.0759588 \frac{1}{\mu a} = \frac{1}{m_1} + \frac{8}{m_2} \quad (13a)$$

$$8.6252476 \frac{1}{\mu a} = \sqrt{\frac{1}{m_1^2} + \frac{9}{m_1m_2} + \frac{64}{m_2^2}} \quad (13b)$$

(13a) and (13b) are squared, and the results are subtracted. The result reads

$$7.9781322 \left(\frac{1}{\mu a} \right)^2 = \frac{7}{m_1m_2} \quad (13c)$$

(13a) and (13c) have the alternative solutions

$$\left. \begin{aligned} m_1 &= 0.8692291\mu a \\ m_2 &= 1.0093982\mu a \end{aligned} \right\} \quad (14a)$$

$$\left. \begin{aligned} m_1 &= 0.1261748\mu a \\ m_2 &= 6.9538330\mu a \end{aligned} \right\} \quad (14b)$$

Both solutions (14a) and (14b) imply that the eigenfrequencies of the discrete system will be identical to the lowest two eigenfrequencies of the continuous system. A simple lumped mass solution to the problem gives the result $m_1 = \mu a$, $m_2 = 2\mu a$. From this it can be concluded that (14a) is the physical meaningful solution, which consequently should be preferred.

Question 2:

The stationary solution to (6) is given as, cf. (3-100), (3-101), (3-102)

$$\mathbf{x}(t) = \mathbf{X} \cos(\omega t) \quad (15)$$

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \mathbf{H}(\omega)\mathbf{F}_0 \quad (16)$$

$$\mathbf{H}(\omega) = (\mathbf{K} - \omega^2 \mathbf{M})^{-1} =$$

$$\frac{4}{3} \frac{a^3}{EI} \frac{1}{\left(1 - \frac{\omega^2}{\omega_1^2}\right) \left(1 - \frac{\omega^2}{\omega_2^2}\right)} \begin{bmatrix} 16 - 9.4210499\alpha & 5 \\ 5 & 2 - 8.1128049\alpha \end{bmatrix} \quad (17)$$

$$\alpha = \frac{\omega^2 \mu a^4}{EI} \quad (18)$$

where ω_1^2 and ω_2^2 are given by (2). From (8) and (16) it then follows

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_1^2}\right) \left(1 - \frac{\omega^2}{\omega_2^2}\right)} \begin{bmatrix} 13.5 - 4.934836\alpha \\ 4.666667 - 8.305967\alpha \end{bmatrix} \frac{F_0 a^3}{EI} \quad (19)$$

Question 3:

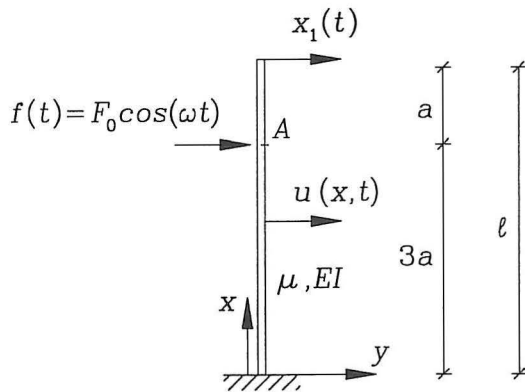


Fig. 2: Continuous beam problem.

A local (x, y) -coordinate system is introduced for the homogeneous cantilever beam as shown in fig. 2. The horizontal displacement from the static equilibrium state is given by the following truncated modal expansion, cf. (4-52)

$$u(x, t) = \sum_{j=1}^2 \Phi^{(j)}(x) q_j(t) \quad (20)$$

The eigenmodes are given as, see (4-43)

$$\begin{aligned} \Phi^{(j)}(x) = & (\cos \lambda_j + \cosh \lambda_j) \left(\sin\left(\lambda_j \frac{x}{l}\right) - \sinh\left(\lambda_j \frac{x}{l}\right) \right) - \\ & (\sin \lambda_j + \sinh \lambda_j) \left(\cos\left(\lambda_j \frac{x}{l}\right) - \cosh\left(\lambda_j \frac{x}{l}\right) \right) \quad , \quad j = 1, 2 \end{aligned} \quad (21)$$

where the eigenvalues λ_j are given by (1). $l = 4a$ signifies the length of the beam, see fig. 2.

The modal coordinates are obtained as the solutions of the decoupled differential equations, see (4-53)

$$\ddot{q}_j + \omega_j^2 q_j = \frac{1}{M_j} F_j(t) \quad , \quad j = 1, 2 \quad (22)$$

The modal mass is given by (4-46)

$$\begin{aligned} M_j = & \int_0^l \mu(x) (\Phi^{(j)}(x))^2 dx = \\ & \mu \cdot 4a \int_0^1 \left((\cos \lambda_j + \cosh \lambda_j) (\sin(\lambda_j \xi) - \sinh(\lambda_j \xi)) - \right. \\ & \left. (\sin \lambda_j + \sinh \lambda_j) (\cos(\lambda_j \xi) - \cosh(\lambda_j \xi)) \right)^2 d\xi = \begin{cases} 17.1242 \cdot 4\mu a & , \quad j = 1 \\ 2877.82 \cdot 4\mu a & , \quad j = 2 \end{cases} \end{aligned} \quad (23)$$

where the information given in the problem text has been utilized. The concentrated load at $x = 3a$ can formally be written as the following load per unit length, cf. (4-68)

$$f_d(x, t) = f(t) \delta(x - 3a) = F_0 \cos(\omega t) \delta(x - 3a) \quad (24)$$

The modal loads then become, see (4-54)

$$F_j(t) = \int_0^l \Phi^{(j)}(x) f_d(x, t) dx = \Phi^{(j)}(3a) F_0 \cos(\omega t) \quad (25)$$

where

$$\Phi^{(j)}(3a) = \begin{cases} 5.443693 & , \quad j = 1 \\ 14.48248 & , \quad j = 2 \end{cases} \quad (26)$$

With $F_j(t) = F_0 \Phi^{(j)}(3a) \cos(\omega t)$, the stationary solution to (22) becomes

$$q_j(t) = \frac{\Phi^{(j)}(3a)F_0}{M_j(\omega_j^2 - \omega^2)} \cos(\omega t) \quad , \quad j = 1, 2 \quad (27)$$

The stationary displacement response of the free end of the beam then follows from (20) for $x = 4a$ and (27)

$$x_1(t) = X_1 \cos(\omega t) \quad (28)$$

$$X_1 = \sum_{j=1}^2 \frac{\Phi^{(j)}(3a)\Phi^{(j)}(4a)}{M_j(\omega_j^2 - \omega^2)} F_0 \quad (29)$$

where

$$\Phi^{(j)}(4a) = \begin{cases} 8.276268 & , \quad j = 1 \\ -107.2906 & , \quad j = 2 \end{cases} \quad (30)$$

Inserting (2), (23), (26), and (30) into (29), the following result may be obtained

$$X_1 = \frac{1}{\left(1 - \frac{\omega^2}{\omega_1^2}\right) \left(1 - \frac{\omega^2}{\omega_2^2}\right)} (13.549425 - 5.707879\alpha) \frac{F_0 a^3}{EI} \quad (31)$$

where the frequency parameter α is defined in eq. (18) of problem (1). (31) should be compared with the first equation (19) for the discrete system with the same circular eigenfrequencies. The expression deviates due to different numerators $N_1(\omega) = 13.500000 - 4.934836\alpha$ and $N_2(\omega) = 13.549425 - 5.707879\alpha$. Below in table 1 the values of $N_1(\omega)$ and $N_2(\omega)$ have been indicated as functions of the circular frequency ω

ω	$N_1(\omega)$	$N_2(\omega)$
0	13.500	13.549
0.7 ω_1	13.383	13.414
1.0 ω_1	13.262	13.274
2.0 ω_1	12.547	12.446
0.7 ω_2	8.914	8.245
1.0 ω_2	4.141	2.724
2.0 ω_2	-23.936	-29.752

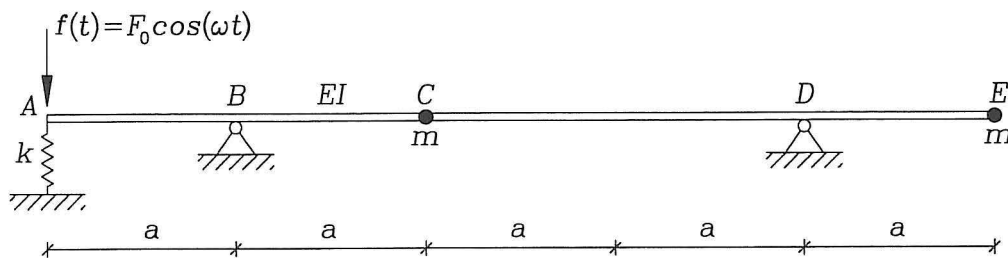
Table 1: Numerators $N_1(\omega)$ and $N_2(\omega)$ in eqs. (19) and (31) as a function of the circular frequency ω .

As seen the agreement is excellent for $\omega \in [0, 2.0\omega_1]$ whereas the difference is much larger for $\omega \in [0.7\omega_2, 2.0\omega_2]$.

2.6 September 28, 1989

Duration: 4h

PROBLEM 1



The figure shows a continuous horizontal plane massless Bernoulli-Euler beam $ABCDE$. The sub-beams AB , BC , DE have the length a , whereas CD has the length $2a$. All sub-beams have a constant bending stiffness EI . The beam is simply supported at points B and D as shown in the figure. At point C and at the free end concentrated masses of magnitude m are placed. At the free end at point A a vertical linear elastic spring with the spring constant k is applied to the beam. Moreover, a vertical harmonic force $f(t) = F_0 \cos(\omega t)$ is acting at point A . The beam is assumed to be infinitely stiff against axial elongations. Only small vibrations from the static equilibrium state are considered.

Question 1 (33%, $\mu = 18.7\%$)

Let $k = \frac{3}{4} \frac{EI}{a^3}$ and $\omega = \sqrt{\frac{EI}{ma^3}}$. Determine the stationary displacement response of the masses from the static equilibrium state, when the dynamic load has been acting for such a period of time that the response from the initial conditions has been dissipated. Otherwise, the effect of any damping mechanism acting on the system is ignored.

SOLUTIONS

PROBLEM 1

Question 1:

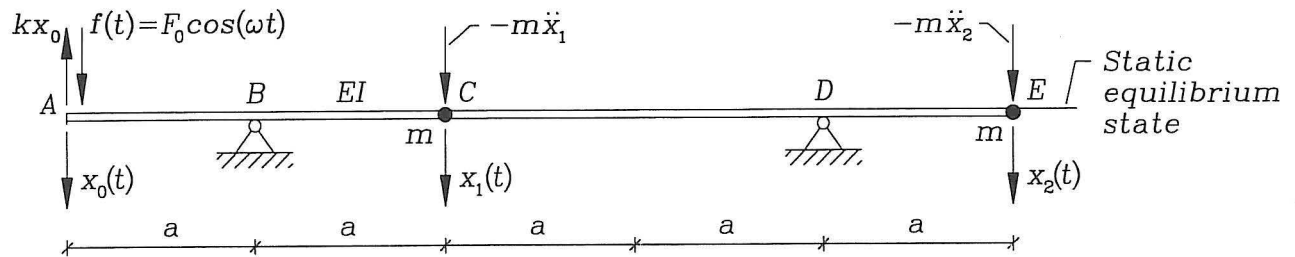


Fig. 1: Forces on a free beam.

The beam is massless. Hence, the system has but 2 degrees of freedom, which are selected as the vertical displacements $x_1(t)$ and $x_2(t)$ of point C and point E from the static equilibrium state with signs as defined in fig. 1. The vertical displacement of point A at the indirectly acting external force from the static equilibrium state $x_0(t)$ is introduced as an auxiliary degree of freedom. The beam is cut free from the spring, and the spring force $kx_0(t)$ is applied as an external force with a sign as shown in fig. 1. Finally, the inertial forces $-m\ddot{x}_1$ and $-m\ddot{x}_2$ are applied as external forces at the points C and E according to d'Alembert's principle. The equations of motion then read, cf. (3-342), (3-343)

$$x_0(t) = \delta_{00}(f(t) - kx_0) + \delta_{01}(-m\ddot{x}_1) + \delta_{02}(-m\ddot{x}_2) \quad (1)$$

$$x_1(t) = \delta_{01}(f(t) - kx_0) + \delta_{11}(-m\ddot{x}_1) + \delta_{12}(-m\ddot{x}_2) \quad (2a)$$

$$x_2(t) = \delta_{02}(f(t) - kx_0) + \delta_{21}(-m\ddot{x}_1) + \delta_{22}(-m\ddot{x}_2) \quad (2b)$$

The flexibility coefficients are given as, cf. (B-1), (B-2), (B-3), (B-4)

$$\delta_{00} = \frac{4}{3} \frac{a^3}{EI} \quad (3)$$

$$\mathbf{d}_0 = \begin{bmatrix} \delta_{01} \\ \delta_{02} \end{bmatrix} = \frac{1}{18} \frac{a^3}{EI} \begin{bmatrix} -10 \\ 9 \end{bmatrix} \quad (4)$$

$$\mathbf{D} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \frac{4}{9} \frac{a^3}{EI} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \quad (5)$$

Notice that \mathbf{D} signifies the flexibility matrix of the system, only if the spring is absent. From (1) it follows that

$$x_0(t) = \frac{1}{1 + k\delta_{00}} \left(\delta_{00}f(t) + \delta_{01}(-m\ddot{x}_1) + \delta_{02}(-m\ddot{x}_2) \right) \quad (6)$$

By inserting (6) into (2) the following equations of motion are obtained

$$\left. \begin{aligned} x_1(t) &= \delta_{01}f(t) - \frac{k\delta_{01}}{1 + k\delta_{00}} \left(\delta_{00}f(t) + \delta_{01}(-m\ddot{x}_1) + \delta_{02}(-m\ddot{x}_2) \right) + \\ &\quad \delta_{11}(-m\ddot{x}_1) + \delta_{12}(-m\ddot{x}_2) \\ x_2(t) &= \delta_{02}f(t) - \frac{k\delta_{02}}{1 + k\delta_{00}} \left(\delta_{00}f(t) + \delta_{01}(-m\ddot{x}_1) + \delta_{02}(-m\ddot{x}_2) \right) + \\ &\quad \delta_{21}(-m\ddot{x}_1) + \delta_{22}(-m\ddot{x}_2) \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -m \begin{bmatrix} \delta_{11} - \frac{k\delta_{01}^2}{1 + k\delta_{00}} & \delta_{12} - \frac{k\delta_{01}\delta_{02}}{1 + k\delta_{00}} \\ \delta_{21} - \frac{k\delta_{01}\delta_{02}}{1 + k\delta_{00}} & \delta_{22} - \frac{k\delta_{02}^2}{1 + k\delta_{00}} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} +$$

$$\begin{bmatrix} \delta_{01} - \frac{k\delta_{00}\delta_{01}}{1 + k\delta_{00}} \\ \delta_{02} - \frac{k\delta_{00}\delta_{02}}{1 + k\delta_{00}} \end{bmatrix} F_0 \cos(\omega t) \quad (7)$$

By using (3), (4), (5), and $k = \frac{3}{4} \frac{EI}{a^3} = \frac{1}{\delta_{00}}$, (7) can be written

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{K}\mathbf{F}_0 \cos(\omega t) = \mathbf{F} \cos(\omega t) \quad (8)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} \delta_{11} - \frac{k\delta_{01}^2}{1 + k\delta_{00}} & \delta_{12} - \frac{k\delta_{01}\delta_{02}}{1 + k\delta_{00}} \\ \delta_{21} - \frac{k\delta_{01}\delta_{02}}{1 + k\delta_{00}} & \delta_{22} - \frac{k\delta_{02}^2}{1 + k\delta_{00}} \end{bmatrix}^{-1} = \frac{1}{252} \frac{EI}{a^3} \begin{bmatrix} 1071 & 294 \\ 294 & 284 \end{bmatrix} \quad (9)$$

$$\mathbf{F}_0 = \begin{bmatrix} \delta_{01} - \frac{k\delta_{00}\delta_{01}}{1 + k\delta_{00}} \\ \delta_{02} - \frac{k\delta_{00}\delta_{02}}{1 + k\delta_{00}} \end{bmatrix} F_0 = \frac{1}{36} \begin{bmatrix} -10 \\ 9 \end{bmatrix} \frac{F_0 a^3}{EI} \quad (10)$$

$$\mathbf{F} = \mathbf{K}\mathbf{F}_0 = -\frac{8}{189} \begin{bmatrix} 21 \\ 1 \end{bmatrix} F_0 \quad (11)$$

\mathbf{K} signifies the stiffness matrix of the system with the concentrated spring present. With $\mathbf{f}(t) = \mathbf{F} \cos(\omega t)$, the stationary response of the masses is given as, cf. (3-100), (3-101)

$$\mathbf{x}(t) = \mathbf{X} \cos(\omega t) \quad (12)$$

$$\mathbf{X} = \mathbf{H}(\omega)\mathbf{F} \quad (13)$$

$$\mathbf{H}(\omega) = (\mathbf{K} - \omega^2 \mathbf{M})^{-1} = \frac{252}{D} \begin{bmatrix} 284 - \lambda & -294 \\ -294 & 1071 - \lambda \end{bmatrix} \frac{a^3}{EI} \quad (14)$$

$$D = (1071 - \lambda)(284 - \lambda) - 294 \cdot 294 \quad (15)$$

$$\lambda = 252 \frac{\omega^2 m a^3}{EI} \quad (16)$$

$\mathbf{x}(t) = \mathbf{X} \cos(\omega t)$ is in phase with the excitation $\mathbf{f}(t) = \mathbf{F} \cos(\omega t)$, and \mathbf{X} is real, because the structural system is free of damping. By inserting (11) and (14) into (13), the following solution is obtained for the amplitude vector

$$\mathbf{X} = \frac{32}{3} \frac{1}{\lambda^2 - 1355\lambda + 217728} \begin{bmatrix} 21\lambda - 5670 \\ \lambda + 5103 \end{bmatrix} \frac{F_0 a^3}{EI} \quad (17)$$

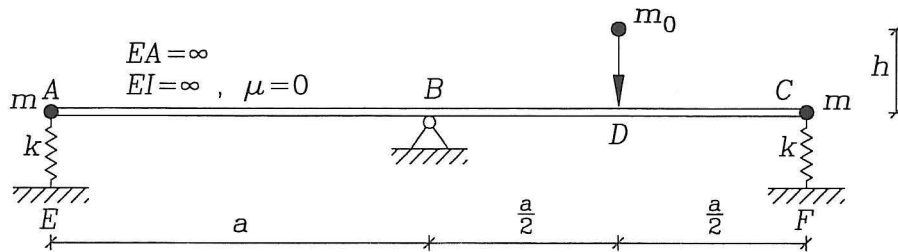
where λ is given by (16). With $\omega = \sqrt{\frac{EI}{ma^3}}$ follows that $\lambda = 252$. (17) then attains the value

$$\mathbf{X} = \frac{8}{717} \begin{bmatrix} 6 \\ -85 \end{bmatrix} \frac{F_0 a^3}{EI} \quad (18)$$

2.7 September 5, 1990

Duration: 3.5 h

PROBLEM 1



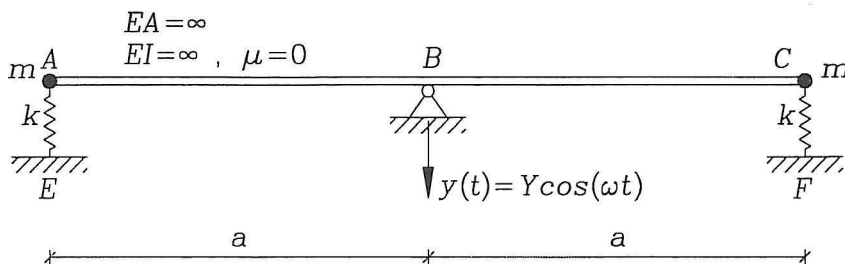
The figure shows a horizontal plane beam ABC at rest. The beam is simply supported at point B . Both sub-beams AB and BC are massless, infinitely stiff against bending and axial elongations, and have the length a . At the free ends A and C concentrated masses of magnitude m and vertical linear elastic springs with the spring constant k are applied. The other ends of the springs are fixed to the support at the points E and F .

A concentrated mass m_0 is placed at rest at the height h above the midpoint D of the beam BC . Next, the mass m_0 is performing a free fall and hits the beam at the point D . The air resistance during the fall is ignored. The acceleration of gravity is g .

Question 1 (25%, $\mu = 10.4\%$)

The impact of the mass m_0 is assumed to be completely inelastic (i.e. the mass is fixed to the beam at point D after the impact). Determine the motion of the system after the impact. Only small vibrations are considered, and the influence of any damping mechanism is ignored.

PROBLEM 2



The system of problem 1 is considered again. However, the beam ABC is now a plane massless Bernoulli-Euler beam with a constant bending stiffness EI . As before, the beam is assumed to be infinitely stiff against axial elongations, and it has the same boundary conditions as in problem 1.

Question 1 (15%, $\mu = 8.0\%$)

Determine the undamped circular eigenfrequencies and the eigenmodes of the system, and make a sketch of the eigenmodes.

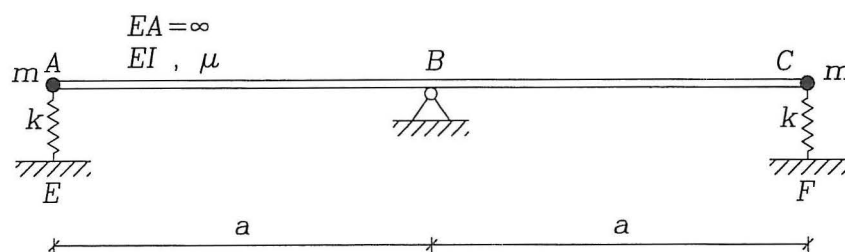
Question 2 (15%, $\mu = 4.8\%$)

The fixed simple support at point B performs a harmonic vertical displacement $y(t) = Y \cos(\omega t)$ with a sign as defined in the figure. The support points E and F of the springs remain at rest. Determine the stationary displacement response of the masses from the static equilibrium state, assuming that the motion of point D has been acting for such period of time that the response from the initial conditions has been dissipated. Otherwise, the effect of any damping mechanism acting on the system is ignored.

Question 3 (10%, $\mu = 1.5\%$)

Assume that the support points E and F of the springs are excited by the same vertical displacement $y(t) = Y \cos(\omega t)$ as the fixed simple support at point B . Determine the stationary displacement of the masses at the points A and C with the same assumptions as stated in problem 2.

PROBLEM 3



The system of problem 2 is considered again. However, the beam ABC is no longer assumed to be massless, but has the constant mass per unit length μ .

Question 1 (35%, $\mu = 18.8\%$)

Formulate the frequency condition for the determination of the undamped circular eigenfrequencies of the structure (no numerical solution of the frequency condition is required).

SOLUTIONS

PROBLEM 1

Question 1:

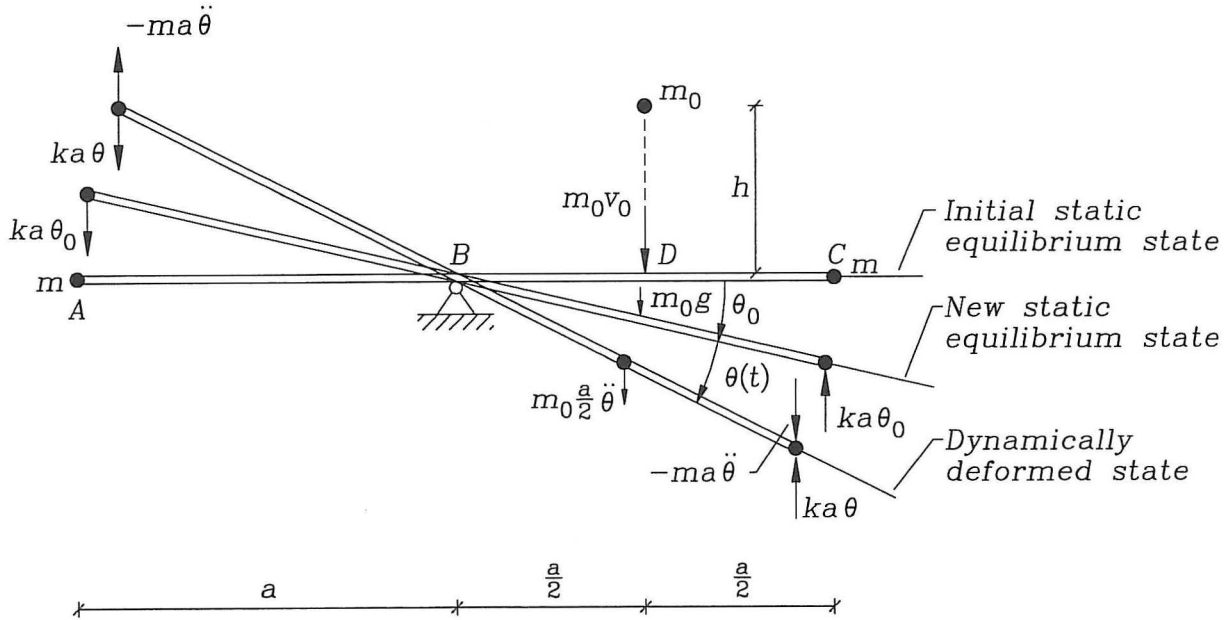


Fig. 1: Definition of equilibrium states and forces on a free beam.

The initial static equilibrium is the horizontal rectilinear state of the beam ABC , before the mass m_0 hits the beam at point D . After m_0 is fixed to the beam the gravity force m_0g implies a new static equilibrium state, which the beam eventually attains, when the eigenvibrations caused by the impact have been dissipated.

Because the beam is infinitely stiff, the beam has but a single degree of freedom which is selected as the clockwise rotation angle of the support point B . The new static equilibrium state is specified by the rotation angle θ_0 from the initial static equilibrium state, see fig. 1. In the new static equilibrium state the spring forces have increased with the magnitude $ka\theta_0$ with signs as defined in fig. 1. These forces must balance the gravity load m_0g . Expressing the moment equilibrium around point B one has

$$m_0g \cdot \frac{a}{2} = 2ka\theta_0 \cdot a \Rightarrow$$

$$\theta_0 = \frac{m_0g}{4ka} \quad (1)$$

After the impact the system performs undamped eigenvibrations. The dynamically deformed state is measured by the angle $\theta(t)$ from the new static equilibrium state. Then all static load disappears from the equation of motion. The beam is cut free from

the springs and the increase of the spring forces $ka\theta(t)$ are applied as external loads at the points A and C with signs as defined in fig. 1. According to d'Alembert's principle, the inertial loads $-ma\ddot{\theta}$ are applied at the points A and C and the inertial load $-m_0\frac{a}{2}\ddot{\theta}$ is applied at point D with signs as defined in fig. 1. The equation of motion is then obtained expressing the moment equilibrium around point B

$$-2ma\ddot{\theta} \cdot a - m_0\frac{a}{2}\ddot{\theta} \cdot \frac{a}{2} = 2ka\theta \cdot a \quad \Rightarrow$$

$$\ddot{\theta} + \omega_0^2\theta = 0 \quad , \quad t > 0 \quad (2)$$

$$\omega_0 = \sqrt{\frac{8k}{m_0 + 8m}} \quad (3)$$

ω_0 is the undamped circular eigenfrequency of the system. In order to solve (2), the initial displacement $\theta(0^+)$ and the initial velocity $\dot{\theta}(0^+)$ at the time $t = 0^+$ immediately after the impact must be determined.

Because the impact is of infinitely short duration the beam has not moved at the end of the impact, and is still in the initial equilibrium state. Hence

$$\theta(0^+) = -\theta_0 = -\frac{m_0g}{4ka} \quad (4)$$

The velocity of the mass m_0 immediately before the impact is

$$v_0 = \sqrt{2gh} \quad (5)$$

The momentum of m_0 (and of the entire system) before the impact is then m_0v_0 . Immediately after the impact the spring forces have not changed, because the system has not moved. Hence, the only external force on the system is the reaction force at point B . If the equation of moment of momentum is formulated around point B , this reaction force disappears from the equation. Expressing that the moment of momentum around point B of all masses before and after the impact is identical, one then has, see figure 1

$$m_0v_0 \cdot \frac{a}{2} = 2ma\dot{\theta}(0^+) \cdot a + m_0\frac{a}{2}\dot{\theta}(0^+) \cdot \frac{a}{2} \quad \Rightarrow$$

$$\dot{\theta}(0^+) = \frac{2m_0}{m_0 + 8m} \frac{v_0}{a} = \frac{2m_0}{m_0 + 8m} \frac{\sqrt{2gh}}{a} \quad (6)$$

The motion of the system from the new static equilibrium state then becomes, see (2-8)

$$\theta(t) = \theta(0^+) \cos(\omega_0 t) + \frac{\dot{\theta}(0^+)}{\omega_0} \sin(\omega_0 t) =$$

$$-\frac{m_0 g}{4ka} \cos\left(\sqrt{\frac{8k}{m_0 + 8m}}t\right) + \frac{m_0}{\sqrt{k(m_0 + 8m)}} \frac{\sqrt{gh}}{a} \sin\left(\sqrt{\frac{8k}{m_0 + 8m}}t\right) \quad (7)$$

PROBLEM 2

Question 1:

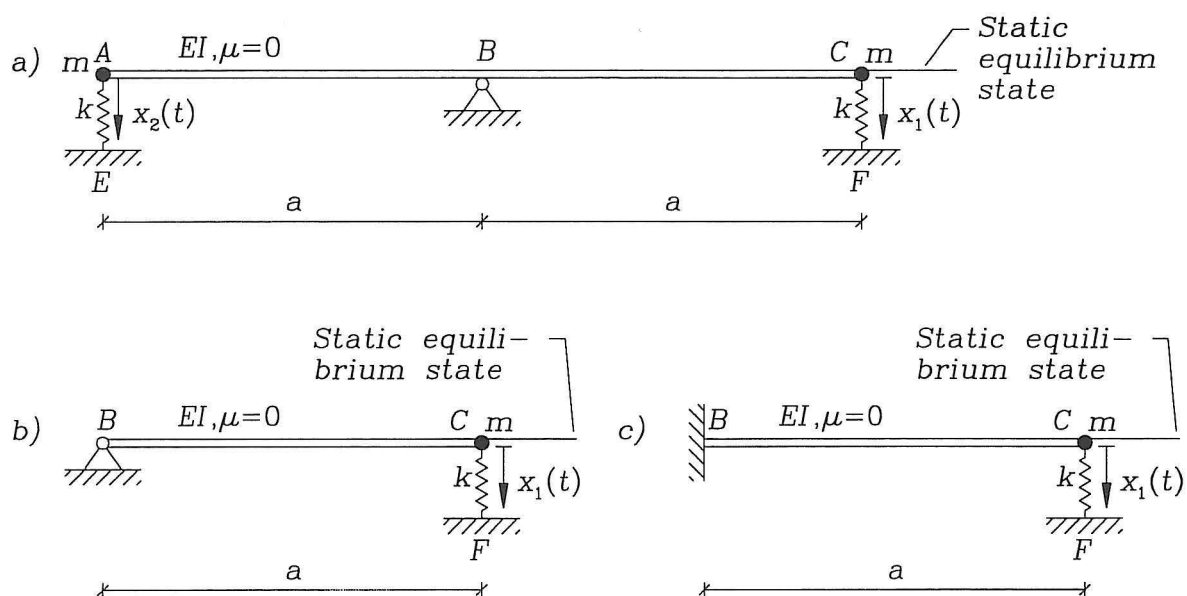


Fig. 1: 2 degrees-of-freedom system. b) Equivalent system for anti-symmetric eigenvibrations. c) Equivalent system for symmetric eigenvibrations.

The beam is massless. Hence, the system has but 2 degrees of freedom, which are selected as the vertical displacements $x_1(t)$ and $x_2(t)$ of the points C and A from the horizontal static equilibrium state with signs as defined in fig. 1a.

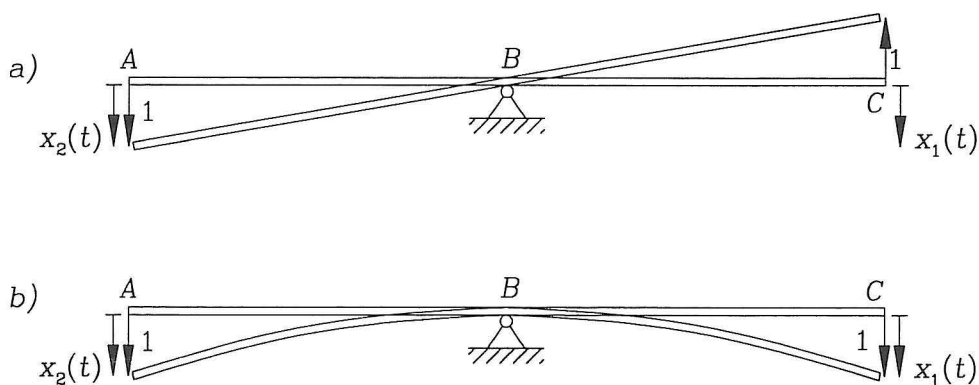


Fig. 2: Sketch of eigenmodes. a) 1st eigenmode. b) 2nd eigenmode.

The anti-symmetric eigenvibrations have zero displacement and zero bending moment at point B . The equivalent single degree-of-freedom system has been shown in fig. 1b. The beam remains rectilinear under the eigenvibrations as shown in fig. 2a. The circular eigenfrequency is given as

$$\omega_1 = \sqrt{\frac{k}{m}} \quad (1)$$

The corresponding eigenmode becomes, see fig. 2a

$$\Phi^{(1)} = \begin{bmatrix} \Phi_1^{(1)} \\ \Phi_2^{(1)} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad (2)$$

The symmetric eigenvibrations have zero displacement and zero slope at point B . The equivalent single degree of freedom has been shown in fig. 1c. In this case the beam provides the stiffness contribution $3\frac{EI}{a^3}$ to the spring stiffness k . Hence, the circular eigenfrequency becomes

$$\omega_2 = \sqrt{\frac{k + 3\frac{EI}{a^3}}{m}} \quad (3)$$

The corresponding eigenmode becomes, see fig. 2b

$$\Phi^{(2)} = \begin{bmatrix} \Phi_1^{(2)} \\ \Phi_2^{(2)} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (4)$$

Question 2:

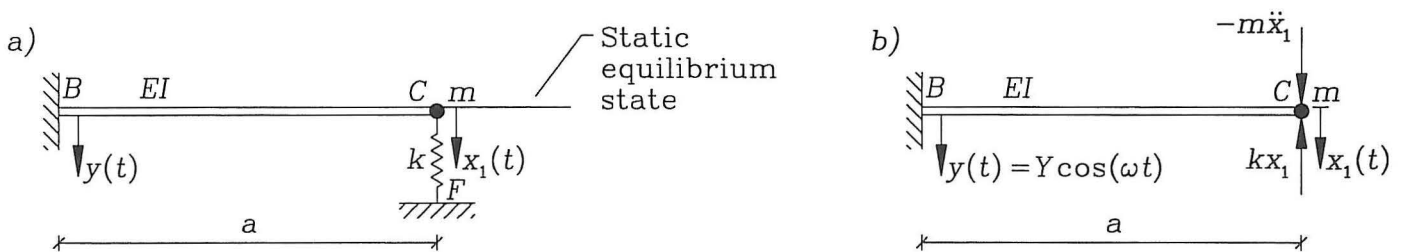


Fig. 3: a) Equivalent system for the analysis of vibrations due to motion of the support at point B . b) Forces on a free beam.

The forced vibrations due to the motion of the support at point B must be symmetric around point B . Hence, these vibrations can be analysed by means of the equivalent single degree of freedom shown in fig. 3a. The stiffbody motion is $x_1^{(0)}(t) = y(t)$. The beam is cut free from the spring, and the spring force $kx_1(t)$ is applied as an external

force with sign as shown in fig. 3b. Further, the inertial load $-m\ddot{x}_1$ is applied as an external load. The equation of motion then becomes, cf. (3-328)

$$x_1(t) = y(t) + \delta_{11}(-m\ddot{x}_1 - kx_1) \quad (5)$$

$$\delta_{11} = \frac{1}{3} \frac{a^3}{EI} \quad (6)$$

(5) can be written

$$\ddot{x}_1 + \omega_2^2 x_1 = 3 \frac{EI}{ma^3} Y \cos(\omega t) \quad (7)$$

where ω_2 is given by (3). The stationary response of (7) becomes

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} X_1 \cos(\omega t) \quad (8)$$

$$X_1 = \frac{1}{\omega_2^2 - \omega^2} 3 \frac{EI}{ma^3} Y = \frac{\omega_2^2 - \omega_1^2}{\omega_2^2 - \omega^2} Y \quad (9)$$

where $3 \frac{EI}{ma^3} = \omega_2^2 - \omega_1^2$ has been introduced, cf. (1) and (3). $\mathbf{x}(t)$ is in phase with the excitation $y(t) = Y \cos(\omega t)$, because the structural system is free of damping.

Question 3

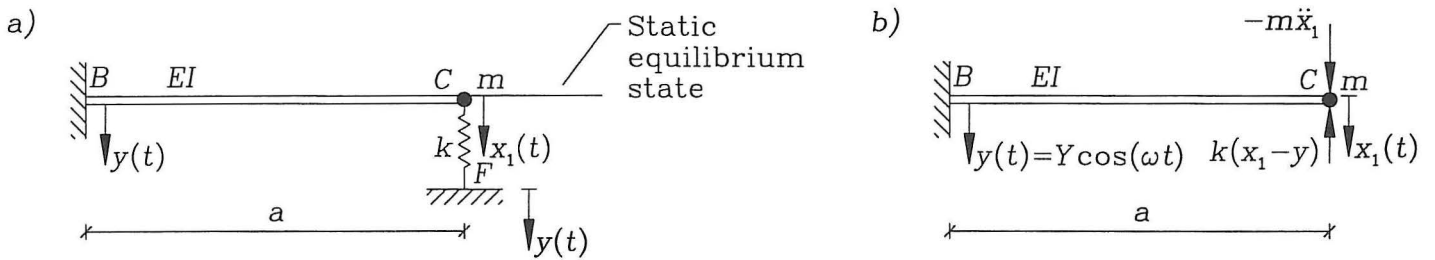


Fig. 4: a) Equivalent system for the analysis of vibrations due to coherent motion of the supports at points B and F . b) Forces on a free beam.

The excitation and hence the response of the masses are still symmetric around point B . The response can then be analysed by means of the equivalent single degree-of-freedom system shown in fig. 4a. The only difference to the system in fig. 3a is that the spring force in the present case becomes $k(x_1(t) - y(t))$, see fig. 4b. Equation (5) is then replaced by the following modified equation of motion

$$x_1(t) = y(t) + \delta_{11}(-m\ddot{x}_1 - k(x_1 - y(t))) \Rightarrow$$

$$\ddot{x}_1 + \omega_2^2 = \omega_2^2 Y \cos(\omega t) \quad (10)$$

The stationary response of (10) becomes

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} X_1 \cos(\omega t) \quad (11)$$

$$X_1 = \frac{\omega_2^2}{\omega_2^2 - \omega^2} Y \quad (12)$$

PROBLEM 3

Question 1:

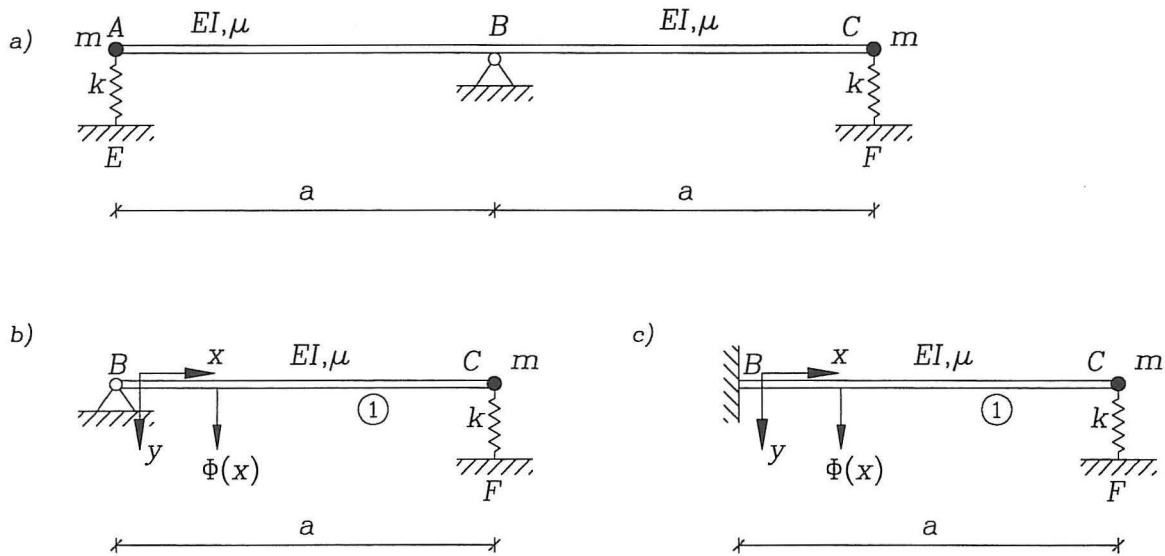


Fig. 1: Symmetric, continuous, homogeneous beam. b) Equivalent system for anti-symmetric eigenvibrations. c) Equivalent system for symmetric eigenvibrations.

The structure shown in fig. 1a is mechanically and geometrically symmetric around point B . Hence, the eigenvibrations separate into anti-symmetric and symmetric eigenvibrations. The equivalent systems for the analysis of anti-symmetric eigenvibrations and symmetric eigenvibrations have been shown in fig. 1b and fig. 1c, respectively. As seen a local (x, y) -coordinate system has been defined for both reduced systems.

Because the beam AB has constant bending stiffness EI , constant mass per unit length μ , and no normal force N is present, the eigenmodes are given as, cf. (4-18), (4-19)

$$\Phi(x) = A \sin\left(\lambda \frac{x}{a}\right) + B \cos\left(\lambda \frac{x}{a}\right) + C \sinh\left(\lambda \frac{x}{a}\right) + D \cosh\left(\lambda \frac{x}{a}\right) \quad (1)$$

$$\lambda^4 = \frac{\mu \omega^2 a^4}{EI} \quad (2)$$

First, anti-symmetric eigenvibrations are analysed. The boundary conditions at the fixed simple support at point B ($x = 0$) of the equivalent system becomes, see (4-23)

$$\Phi(0) = \frac{d^2}{dx^2} \Phi(0) = 0 \quad (3)$$

(3) implies that $B = D = 0$, see (4-24). Then, (1) reduces to

$$\Phi(x) = A \sin\left(\lambda \frac{x}{a}\right) + C \sinh\left(\lambda \frac{x}{a}\right) \quad (4)$$

The mechanical boundary conditions at point C ($x = a$) become, see (4-13)

$$\frac{d^2}{dx^2} \Phi(a) = 0 \quad (5)$$

$$EI \frac{d^3}{dx^3} \Phi(a) = (k - \omega^2 m) \Phi(a) \Rightarrow$$

$$a^3 \frac{d^3}{dx^3} \Phi(a) = \left(\kappa - \lambda^4 \frac{m}{\mu a} \right) \Phi(a) \quad (6)$$

$$\kappa = \frac{ka^3}{EI} \quad (7)$$

In the last statement of (6) $\omega^2 = \lambda^4 \frac{EI}{\mu a^4}$ has been introduced, see (2). By inserting (4) into (5) and (6) the following homogeneous equations are obtained for the determination of A and C

$$\left. \begin{aligned} \frac{\lambda^2}{a^2} (-A \sin \lambda + C \sinh \lambda) &= 0 \\ \lambda^3 (-A \cos \lambda + C \cosh \lambda) &= \left(\kappa - \lambda^4 \frac{m}{\mu a} \right) (A \sin \lambda + C \sinh \lambda) \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} -\sin \lambda & \sinh \lambda \\ -\lambda^3 \cos \lambda - \left(\kappa - \lambda^4 \frac{m}{\mu a}\right) \sin \lambda & \lambda^3 \cosh \lambda - \left(\kappa - \lambda^4 \frac{m}{\mu a}\right) \sinh \lambda \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (8)$$

Non-trivial solutions $A \neq 0 \vee C \neq 0$ are obtained if the determinant of (8) is 0. This provides the frequency condition

$$\begin{aligned} \lambda^3 (-\sin \lambda \cosh \lambda + \cos \lambda \sinh \lambda) + 2 \left(\kappa - \lambda^4 \frac{m}{\mu a} \right) \sin \lambda \sinh \lambda &= 0 \Rightarrow \\ \lambda^3 (\tanh \lambda - \tan \lambda) + 2 \left(\kappa - \lambda^4 \frac{m}{\mu a} \right) \tan \lambda \tanh \lambda &= 0 \end{aligned} \quad (9)$$

For $k = 10 \frac{EI}{a^3} \Rightarrow \kappa = 10, m = 2\mu a$, the 5 lowest solutions of (9) become

$$\lambda_j = \begin{cases} 1.43737 & , \quad j = 1 \\ 3.21681 & , \quad j = 3 \\ 6.32133 & , \quad j = 5 \\ 9.45056 & , \quad j = 7 \\ 12.58585 & , \quad j = 9 \end{cases} \quad (10)$$

The difference of (10) and the corresponding results in lecture 10, problem 1 are due to the concentrated mass.

Next, symmetric eigenvibrations are analysed. The geometrical boundary conditions at point B of the equivalent system become, cf. (4-36)

$$\Phi(0) = \frac{d}{dx} \Phi(0) = 0 \quad (11)$$

Inserting (1) into (11) implies $C = -A$ and $D = -B$. Then (1) reduces to, cf. (4-38)

$$\Phi(x) = A \left(\sin \left(\lambda \frac{x}{a} \right) - \sinh \left(\lambda \frac{x}{a} \right) \right) + B \left(\cos \left(\lambda \frac{x}{a} \right) - \cosh \left(\lambda \frac{x}{a} \right) \right) \quad (12)$$

The mechanical boundary conditions at point C ($x = a$) are still given by (5) and (6). Inserting (12) into (5) and (6) the following homogeneous equations are obtained for the determination of A and B

$$\left. \begin{aligned} \frac{\lambda^2}{a^2} \left(-A(\sin \lambda + \sinh \lambda) - B(\cos \lambda + \cosh \lambda) \right) &= 0 \\ \lambda^3 \left(-A(\cos \lambda + \cosh \lambda) + B(\sin \lambda - \sinh \lambda) \right) - \\ \left(\kappa - \lambda^4 \frac{m}{\mu a} \right) \left(A(\sin \lambda - \sinh \lambda) + B(\cos \lambda - \cosh \lambda) \right) &= 0 \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} \sin \lambda + \sinh \lambda & \cos \lambda + \cosh \lambda \\ \lambda^3(\cos \lambda + \cosh \lambda) + \left(\kappa - \lambda^4 \frac{m}{\mu a}\right)(\sin \lambda - \sinh \lambda) & -\lambda^3(\sin \lambda - \sinh \lambda) + \left(\kappa - \lambda^4 \frac{m}{\mu a}\right)(\cos \lambda - \cosh \lambda) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (13)$$

The frequency condition then reads

$$\begin{aligned} & \lambda^3(-\sin^2 \lambda + \sinh^2 \lambda - \cos^2 \lambda - 2 \cos \lambda \cosh \lambda - \cosh^2 \lambda) + \\ & \left(\kappa - \lambda^4 \frac{m}{\mu a}\right) \left((\sin \lambda + \sinh \lambda)(\cos \lambda - \cosh \lambda) - (\sin \lambda - \sinh \lambda)(\cos \lambda + \cosh \lambda) \right) = 0 \Rightarrow \\ & \lambda^3(1 + \cos \lambda \cosh \lambda) + \left(\kappa - \lambda^4 \frac{m}{\mu a}\right) (\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda) = 0 \end{aligned} \quad (14)$$

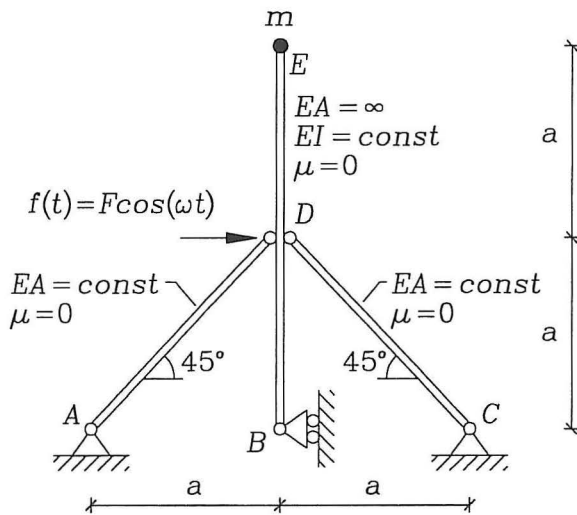
For $k = 10 \frac{EI}{a^3} \Rightarrow \kappa = 10$, $m = 2\mu a$, the 5 lowest solutions of (14) become

$$\lambda_j = \begin{cases} 1.55231 & , \quad j = 2 \\ 3.98363 & , \quad j = 4 \\ 7.10271 & , \quad j = 6 \\ 10.23403 & , \quad j = 8 \\ 13.37012 & , \quad j = 10 \end{cases} \quad (15)$$

As seen from (10) and (15), anti-symmetric and symmetric eigenvibrations are changing.

Duration: 3.5 h

PROBLEM 1



The figure shows a plane idealized structural model of an offshore structure. The mass of the topside is concentrated in a point mass m at point E . Because this mass is dominating, all the beams and bars shown can be considered as massless, i.e. with the mass per unit length $\mu = 0$. Beam BDE is a vertical linear elastic Bernoulli-Euler beam of the length $2a$ with constant bending stiffness EI , and infinite axial stiffness $EA = \infty$. At the midpoint D beam BDE is supported by the bars AD and CD forming an inclination of 45° with the horizontal level. The bars have constant axial stiffness, and are supported, so they can only carry tension and compression. The total load on the structure is given by the horizontal concentrated harmonic varying force $f(t) = F \cos(\omega t)$ acting at point D . Only small horizontal displacements of the mass from its static equilibrium position are considered, and damping is assumed to be negligible. The influence on the displacement of the mass from any normal forces in the static equilibrium state or the deformations due to the shear forces are considered insignificant.

Question 1 (20%, $\mu = 4.8\%$)

Formulate the equation of motion for the displacement of the mass from the static equilibrium state, and determine the undamped circular eigenfrequency of the system.

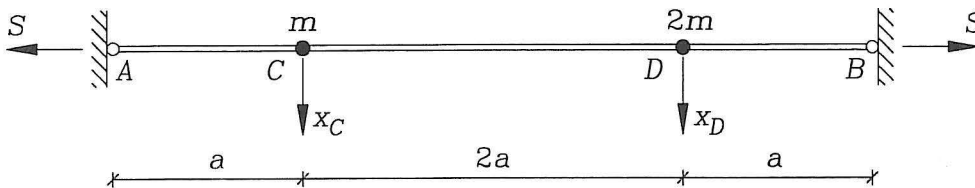
Question 2 (5%, $\mu = 2.2\%$)

Determine the stationary displacement response of the mass, when the dynamic load $f(t) = F \cos(\omega t)$ has been acting for such period of time that the response from the initial conditions has been dissipated.

Question 3 (5%, $\mu = 1.7\%$)

What will happen to the circular eigenfrequency if the bars AD and CD are assumed to be infinitely stiff against axial elongations, i.e. $AE = \infty$?

PROBLEM 2



The figure shows a massless inextensible horizontal string of the length $4a$, which is prestressed with the force S . The string is supported at the points A and B and is completely flexible in bending. At the quarter points C and D at the distance a from A and B point masses of magnitudes m and $2m$ are applied to the string. Only small vertical displacements of the masses are considered, and the horizontal component of the string force is assumed to be constantly equal to S during vibrations. Further, the influence of any damping mechanism is ignored.

Question 1 (25%, $\mu = 13.8\%$)

Determine the circular eigenfrequencies and the eigenmodes of the system. Make a sketch of the eigenmodes.

Question 2 (10%, $\mu = 5.5\%$)

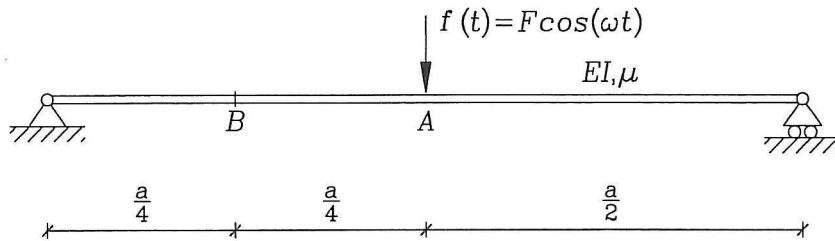
Determine the undamped eigenvibrations of the system in case of the initial conditions

$$x_C(0) = \alpha \quad , \quad x_D(0) = \left(\frac{3 + \sqrt{17}}{4} \right) \alpha$$

$$\dot{x}_C(0) = 0 \quad , \quad \dot{x}_D(0) = 0$$

where $x_C(t)$ and $x_D(t)$ are the vertical displacements of the masses from the static equilibrium state with signs as defined in the figure.

PROBLEM 3



The figure shows a horizontal plane linear elastic simply supported Bernoulli-Euler beam of the length a . The beam has constant bending stiffness EI and constant mass per unit length μ . At the midpoint A the beam is loaded by a vertical harmonic varying concentrated force $f(t) = F \cos(\omega t)$ with amplitude F and circular eigenfrequency ω . Only small vertical vibrations from the static equilibrium state are considered, and the influence of damping is ignored.

Question 1 (25%, $\mu = 15\%$)

Using modal analysis, determine the stationary displacement response of point A , when the displacement response from the initial values has been dissipated. 3 modal coordinates are retained in the truncated modal expansion.

Question 2 (5%, $\mu = 1.3\%$)

How many modal coordinates must be retained in order to obtain an acceptable solution for the displacement response, if the circular frequency of the excitation is given as $\omega = \frac{\pi^2}{2a^2} \sqrt{\frac{EI}{\mu}}$. Motivate the answer.

Question 3 (5%, $\mu = 1.6\%$)

Determine the stationary dynamic bending moment at point B , if 3 modal coordinates are retained in the modal expansion.

SOLUTIONS

PROBLEM 1

Question 1:

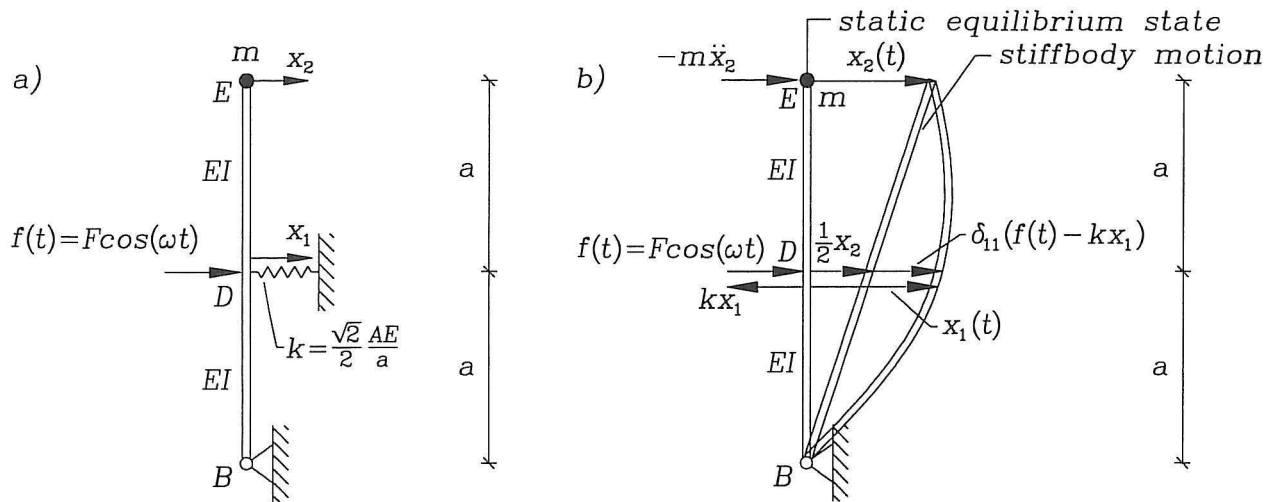


Fig. 1: a) Equivalent system. b) Forces on a free beam and composition of displacements at the spring.

The bars AD and CD can be replaced by a linear elastic spring with the spring constant

$$k = 2 \cdot \frac{AE}{\sqrt{2}a} \cos^2(45^\circ) = \frac{\sqrt{2}}{2} \frac{AE}{a} \quad (1)$$

The resulting equivalent system, which has been shown in fig. 1a, is a special case of the system considered in lecture 6, problem 3. The same approach will be used in this case.

The beam is massless. Hence, the system has but a single degree of freedom, which is selected as the horizontal displacement $x_2(t)$ of the mass from the static equilibrium state with sign as defined in fig. 1a. Besides, an artificial degree of freedom $x_1(t)$ is introduced, specifying the horizontal displacement of point D . The beam is cut free from the spring, and the spring force $kx_1(t)$ is applied as an external force with sign as defined in fig. 1b. Finally, the inertial load $-m\ddot{x}_2(t)$ is applied as an external horizontal load on the point mass with sign as defined in fig. 1b according to d'Alembert's principle. Moment equilibrium around point B then provides the equation of motion

$$-m\ddot{x}_2 \cdot 2a + (f(t) - kx_1)a = 0 \quad (2)$$

The displacement $x_1(t)$ of point D is caused partly by a stiffbody contribution of magnitude $\frac{1}{2}x_2(t)$, and partly by the elastic contribution of magnitude $\delta_{11}(f(t) - kx_1)$ from the external dynamic force and the spring force, see fig. 1b. δ_{11} is the flexibility coefficient, when both ends of the beam are simply supported, as seen from the displacement curve shown in fig. 1b. From (B-1) it follows that

$$\delta_{11} = \frac{1}{6} \frac{a^3}{EI} \quad (3)$$

It then follows that

$$\begin{aligned} x_1(t) &= \frac{1}{2}x_2(t) + \delta_{11}(f(t) - kx_1(t)) \Rightarrow \\ x_1(t) &= \frac{1}{1 + \delta_{11}k} \left(\frac{1}{2}x_2(t) + \delta_{11}f(t) \right) \end{aligned} \quad (4)$$

Inserting (4) into (2) the following equation of motion for the determination of the horizontal motion of the mass from the vertical state of equilibrium is obtained

$$\ddot{x}_2 + \omega_0^2 x_2 = \frac{F_0}{m} \cos(\omega t) \quad (5)$$

$$\omega_0 = \frac{1}{2} \sqrt{\frac{k}{m(1 + k\delta_{11})}} = \sqrt{\frac{\sqrt{2}AE}{8m \left(1 + \frac{\sqrt{2}}{12} \frac{EAa^2}{EI}\right) a}} \quad (6)$$

$$F_0 = \frac{1}{2} \frac{1}{1 + k\delta_{11}} F = \frac{1}{2} \frac{1}{1 + \frac{\sqrt{2}}{12} \frac{EAa^2}{EI}} F \quad (7)$$

ω_0 signifies the undamped circular eigenfrequency of the structure.

Question 2:

The system is undamped. Hence, the stationary displacement response of the mass is in phase with the indirectly acting excitation $f(t) = F \cos(\omega t)$. The stationary response as determined from (5) is then given as

$$x_2(t) = X_2 \cos(\omega t) \quad (8)$$

$$X_2 = \frac{F_0}{m(\omega^2 - \omega_0^2)} = \frac{\omega_0^2}{\omega^2 - \omega_0^2} \frac{2F}{k} \quad (9)$$

$\frac{2F}{k}$ indicates the quasi-static part of the response. Hence, the factor $\omega_0^2/(\omega^2 - \omega_0^2)$ indicates the dynamic amplification factor. k and ω_0 are given by (1) and (6).

Question 3:

In the first statement of (6) the limit passing $k \rightarrow \infty$ is performed. Hence

$$\lim_{k \rightarrow \infty} \omega_0 = \frac{1}{2} \sqrt{\frac{1}{m\delta_{11}}} = \sqrt{\frac{3}{2} \frac{EI}{ma^3}} \quad (10)$$

PROBLEM 2

Question 1:

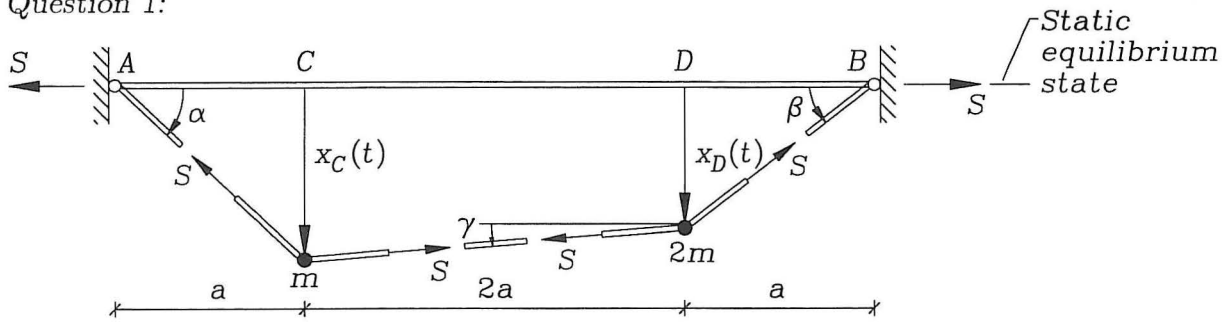


Fig. 1: Free masses during vibrations.

The systems have 2 degrees of freedom, which are selected as the vertical displacements $x_C(t)$ and $x_D(t)$ from the static equilibrium state with signs as defined in fig. 1. The string force has been assumed to be constant equals to S during vibrations. However, the direction of the string force has changed, and the vertical components provide the restoring forces of the masses. The masses are cut free, and the vertical components of S are applied as external loads. By introducing the angles α, β, γ , defined as shown in fig. 1, Newton's 2nd law of motion for each of the two masses provides

$$\left. \begin{aligned} m\ddot{x}_C &= -S \sin \alpha - S \sin \gamma \\ 2m\ddot{x}_D &= S \sin \gamma - S \sin \beta \end{aligned} \right\} \quad (1)$$

Due to the assumption of small vibrations it follows that, see fig. 1

$$\left. \begin{aligned} \sin \alpha &\simeq \frac{x_C}{a} \\ \sin \beta &\simeq \frac{x_D}{a} \\ \sin \gamma &\simeq \frac{x_C - x_D}{2a} \end{aligned} \right\} \quad (2)$$

The linear equations of motion of the system then follow from (1) and (2)

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \quad (3)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_C(t) \\ x_D(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}, \quad \mathbf{K} = \frac{S}{2a} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \quad (4)$$

Question 2:

The circular eigenfrequencies ω_j and the eigenmodes $\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix}$ are determined from the homogeneous linear equations, cf. (3-42)

$$\begin{bmatrix} 3 - \lambda_j & -1 \\ -1 & 3 - 2\lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (5)$$

$$\lambda_j = 2 \frac{\omega_j^2 m a}{S} \quad (6)$$

The characteristic equation becomes

$$(3 - \lambda_j)(3 - 2\lambda_j) - 1 = 0 \Rightarrow$$

$$\lambda_j = \begin{cases} \frac{9 - \sqrt{17}}{4}, & j = 1 \\ \frac{9 + \sqrt{17}}{4}, & j = 2 \end{cases} \quad (7)$$

$$\omega_j = \begin{cases} \sqrt{\frac{9 - \sqrt{17}}{8}} \sqrt{\frac{S}{ma}}, & j = 1 \\ \sqrt{\frac{9 + \sqrt{17}}{8}} \sqrt{\frac{S}{ma}}, & j = 2 \end{cases} \quad (8)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix}, \quad j = 1, 2 \quad (9)$$

The first component $\Phi_1^{(j)}$ is determined from the first equation of (5)

$$(3 - \lambda_j)\Phi_1^{(j)} - 1 \cdot 1 = 0 \Rightarrow$$

$$\Phi_1^{(j)} = \frac{1}{3 - \lambda_j} = \begin{cases} \frac{-3 + \sqrt{17}}{2}, & j = 1 \\ \frac{-3 - \sqrt{17}}{2}, & j = 2 \end{cases} \quad (10)$$

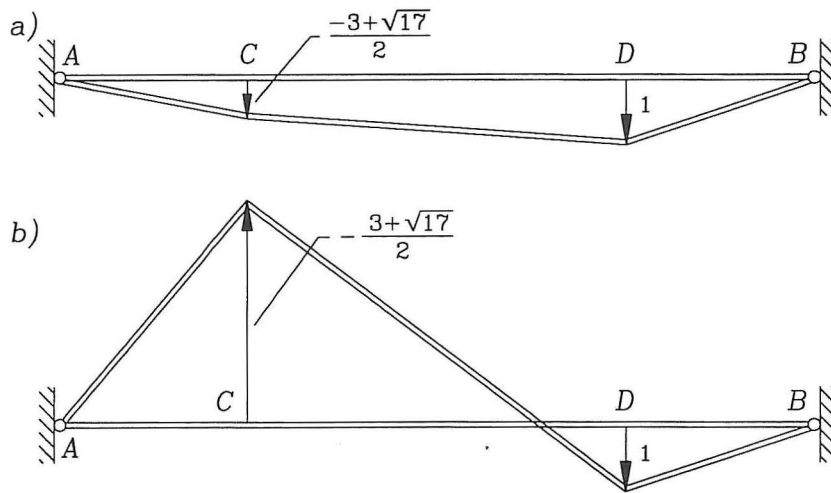


Fig. 2: a) 1st eigenmode. b) 2nd eigenmode.

Question 3:

The eigenvibrations $\mathbf{x}(t) = \begin{bmatrix} x_C(t) \\ x_D(t) \end{bmatrix}$ due to the initial conditions $\mathbf{x}(0) = \begin{bmatrix} 1 \\ \frac{3+\sqrt{17}}{4} \end{bmatrix} \alpha$ and $\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ are given as, cf. (3-68)

$$\mathbf{x}(t) = a_1 \Phi^{(1)} \cos(\omega_1 t) + a_2 \Phi^{(2)} \cos(\omega_2 t) + b_1 \Phi^{(1)} \sin(\omega_1 t) + b_2 \Phi^{(2)} \sin(\omega_2 t) \quad (11)$$

The modal matrix and its inverse becomes, cf. (3-72)

$$\mathbf{P} = \begin{bmatrix} \frac{-3+\sqrt{17}}{2} & -\frac{3+\sqrt{17}}{2} \\ 1 & 1 \end{bmatrix} \Rightarrow$$

$$\mathbf{P}^{-1} = \frac{\sqrt{17}}{17} \begin{bmatrix} 1 & \frac{3+\sqrt{17}}{2} \\ -1 & \frac{-3+\sqrt{17}}{2} \end{bmatrix} \quad (12)$$

The expansion coefficients a_1, a_2, b_1, b_2 are given by (3-70), (3-71)

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{P}^{-1} \mathbf{x}(0) = \begin{bmatrix} \frac{3+\sqrt{17}}{4} \\ 0 \end{bmatrix} \alpha \quad (13)$$

$$\begin{bmatrix} b_1 \omega_1 \\ b_2 \omega_2 \end{bmatrix} = \mathbf{P}^{-1} \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (14)$$

From (11), (13), (14) it follows that

$$\mathbf{x}(t) = \frac{3 + \sqrt{17}}{4} \alpha \begin{bmatrix} \frac{-3 + \sqrt{17}}{2} \\ 1 \end{bmatrix} \cos(\omega_1 t) = \alpha \begin{bmatrix} 1 \\ \frac{3 + \sqrt{17}}{4} \end{bmatrix} \cos(\omega_1 t) \quad (15)$$

$a_2 = 0$, because the initial displacement $\mathbf{x}(0)$ is proportional to the 1st eigenmode $\Phi^{(1)}$.

PROBLEM 3

Question 1:

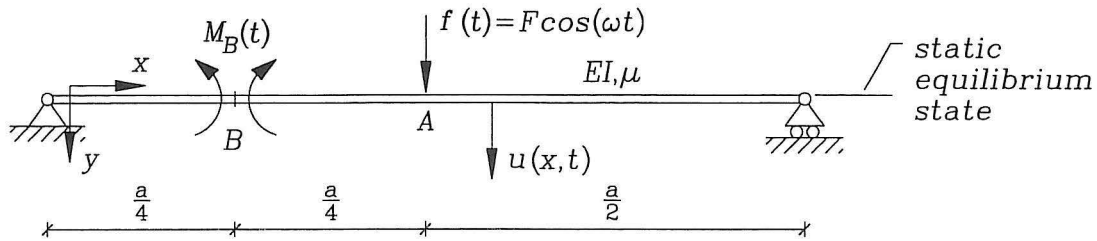


Fig. 1: Continuous simply supported beam. Definition of signs.

A local (x, y) -coordinate system is introduced as shown in fig. 1. The vertical displacement from the static equilibrium state of the homogeneous simply supported beam is given by the following truncated modal expansion, cf. (4-52)

$$u(x, t) = \sum_{j=1}^3 \Phi^{(j)}(x) q_j(t) \quad (1)$$

The eigenmodes are given as, cf. (4-31)

$$\Phi^{(j)}(x) = \sin\left(j\pi \frac{x}{a}\right), \quad j = 1, 2, \dots \quad (2)$$

The undamped circular eigenfrequencies are given as, cf. (4-33)

$$\omega_j = j^2 \pi^2 \sqrt{\frac{EI}{\mu a^4}}, \quad j = 1, 2, \dots \quad (3)$$

With the normalization of the eigenmodes as follows from (2), the modal masses become, cf. (4-65)

$$M_j = \int_0^a \mu (\Phi^{(j)}(x))^2 dx = \frac{1}{2} \mu a, \quad j = 1, 2, \dots \quad (4)$$

Because the system is free of damping, the modal coordinates are obtained as the solutions of the decoupled differential equations, see (4-53)

$$\ddot{q}_j + \omega_j^2 q_j = \frac{1}{M_j} F_j(t) \quad , \quad j = 1, 2, \dots \quad (5)$$

The concentrated load at A ($x = \frac{a}{2}$) can formally be written as the following load per unit length, cf. (4-68)

$$f_d(x, t) = f(t) \delta \left(x - \frac{a}{2} \right) \quad (6)$$

The modal loads then become, see (4-54)

$$F_j(t) = \int_0^a \Phi^{(j)}(x) f_d(x, t) dx = \Phi^{(j)} \left(\frac{a}{2} \right) f(t) = \sin \left(j \frac{\pi}{2} \right) F \cos(\omega t) \quad (7)$$

With $F_j(t)$ given by (7), the stationary solution to (5) becomes

$$q_j(t) = \frac{2 \sin \left(j \frac{\pi}{2} \right)}{\mu a \omega_j^2 - \omega^2} F \cos(\omega t) \quad , \quad j = 1, 2, \dots \quad (8)$$

Inserting (2) and (8) into (1), the stationary displacement response $u_A(t) = u \left(\frac{a}{2}, t \right)$ becomes

$$u_A(t) = \left(\sum_{j=1}^3 \frac{\sin^2 \left(j \frac{\pi}{2} \right)}{\omega_j^2 - \omega^2} \right) \frac{2F}{\mu a} \cos(\omega t) = \left(\frac{1}{\omega_1^2 - \omega^2} + \frac{1}{3^4 \omega_1^2 - \omega^2} \right) \frac{2F}{\mu a} \cos(\omega t) \quad (9)$$

where ω_1^2 is given by (3).

Question 2:

The next non-zero term in the expansion within the parenthesis of (9) is given as $\frac{1}{5^4 \omega_1^2 - \omega^2}$. With $\omega = \frac{\pi^2}{2a^2} \sqrt{\frac{EI}{\mu}} = \frac{1}{2} \omega_1$, the relative magnitude of the terms in the expansion becomes

$$\left. \begin{aligned} \frac{\omega_1^2 - \omega^2}{3^4 \omega_1^2 - \omega^2} &= 0.00929 \\ \frac{\omega_1^2 - \omega^2}{5^4 \omega_1^2 - \omega^2} &= 0.00120 \end{aligned} \right\} \quad (10)$$

Truncation after 3 modal coordinates introduces errors significantly less than 1%, which is acceptable. Hence, 3 modal coordinates should be retained in the expansion.

Question 3:

The sign of the dynamic bending moment $M_B(t)$ at point B ($x = \frac{a}{4}$) is defined in fig.

1. This is given as, cf. (4-61), (2), (8)

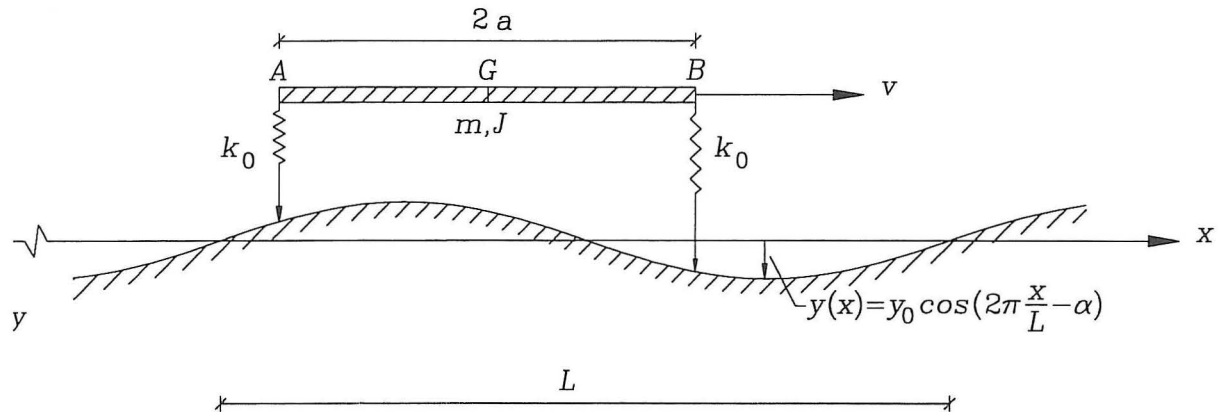
$$\begin{aligned}
 M_B(t) &= -EI \frac{\partial^2 u(\frac{a}{4}, t)}{\partial x^2} = EI \frac{\pi^2}{a^2} \sum_{j=1}^3 j^2 \sin\left(j\pi \frac{1}{4}\right) q_j(t) = \\
 &\pi^4 \frac{EI}{\mu a^4} \left(\frac{\sin\left(\frac{\pi}{4}\right)}{\omega_1^2 - \omega^2} - \frac{3^2 \sin\left(\frac{3\pi}{4}\right)}{3^4 \omega_1^2 - \omega^2} \right) \frac{2}{\pi^2} F a \cos(\omega t) = \\
 &\left(\frac{\omega_1^2}{\omega_1^2 - \omega^2} - \frac{9\omega_1^2}{81\omega_1^2 - \omega^2} \right) \frac{\sqrt{2}}{\pi^2} F a \cos(\omega t)
 \end{aligned} \tag{11}$$

The quasi-static response ($\omega = 0$) of (11) is $M_B = \frac{8}{9} \frac{\sqrt{2}}{\pi^2} F a = 0.12737$, whereas the exact solution is $M_B = \frac{1}{8} F a = 0.12500$. The error is 1.9%.

2.9 June 21, 1991

Duration: 4h

PROBLEM 1



The vehicle shown in the figure consists of a plane, infinitely stiff beam AB of the length $2a$ with a constant mass distribution. The total mass of the beam is m , and the moment of mass inertia around a horizontal line through the centre of gravity is J . At the end points A and B the vehicle is suspended on the vertical linear elastic springs with the spring constant k . The motion from any initial conditions is assumed to be dissipated. Otherwise, the effect of any damping mechanism in the spring or the beam is ignored.

The vehicle is moving with the constant velocity v on a rough road. Relative to a mean position the profile of the road is approximated by a sine wave

$$y(x) = y_0 \cos\left(2\pi \frac{x}{L} - \alpha\right)$$

where x is a horizontal, and y is a vertical coordinate defined as shown in the figure. y_0 is the amplitude, L is the wave length and α is a phase. The amplitude y_0 as well as the vertical displacements of the vehicle is assumed to be small compared to the wave length L and the length $2a$ of the beam AB . The springs are assumed to be in permanent contact with the surface of the road.

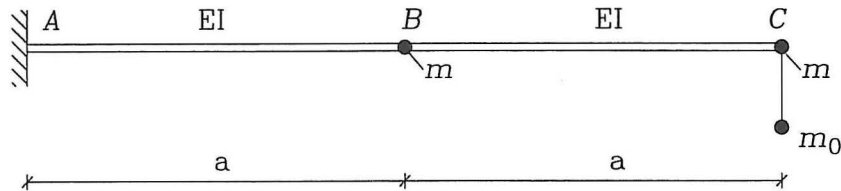
Question 1 (20%, $\mu = 7.1\%$)

Determine the speeds of the vehicle, which should be avoided.

Question 2 (10% , $\mu = 2.0\%$)

Determine the stationary vertical motion for a passenger, who is placed at the centre of gravity G , if $a = \frac{L}{4}$.

PROBLEM 2



The beam AC in the figure is a rectilinear horizontal plane massless Bernoulli-Euler beam with the constant bending stiffness EI . The beam is fixed at point A and free at point C , and the length is $2a$. The beam is considered infinitely stiff against axial deformations.

At the point C and at the midpoint B point masses of magnitude m are applied. Besides, at point C a perfectly flexible massless string is fixed, which is supporting a point mass of magnitude m_0 . The acceleration of gravity is g .

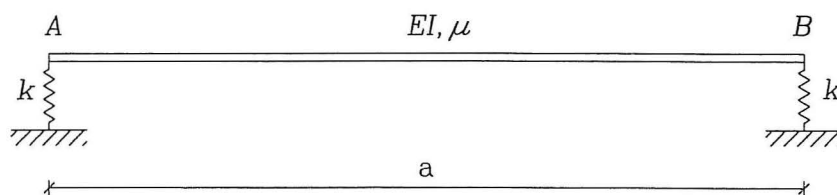
Question 1 (30% , $\mu = 16.6\%$)

At the time $t = 0$ the string at point C is cut. Determine the succeeding motion of the masses. Only small vertical vibrations of the masses are considered, and the effect of any damping mechanism inside the beam AC or from the surroundings is ignored.

Question 2 (10% , $\mu = 5.0\%$)

Determine a possible damping matrix after the string at point C has been cut, if it is known that the remainder system has the modal damping ratios ζ_1 and ζ_2 in the two lowest eigenvibrations.

PROBLEM 3



The beam AB in the figure is a horizontal plane Bernoulli-Euler beam of the length a . The beam has the constant bending stiffness EI , the constant mass per unit length μ , and the normal force in the static equilibrium state is equal to 0. The beam is supported at the end points by two vertical linear elastic springs both with the spring constant k . The beam is considered infinitely stiff against axial elongations. Only small vertical vibrations in the plane of the structure are considered.

Question 1 (30%, $\mu = 20.5\%$)

Formulate the frequency condition for the determination of the undamped circular eigenfrequencies of the beam.

SOLUTIONS

PROBLEM 1

Question 1:

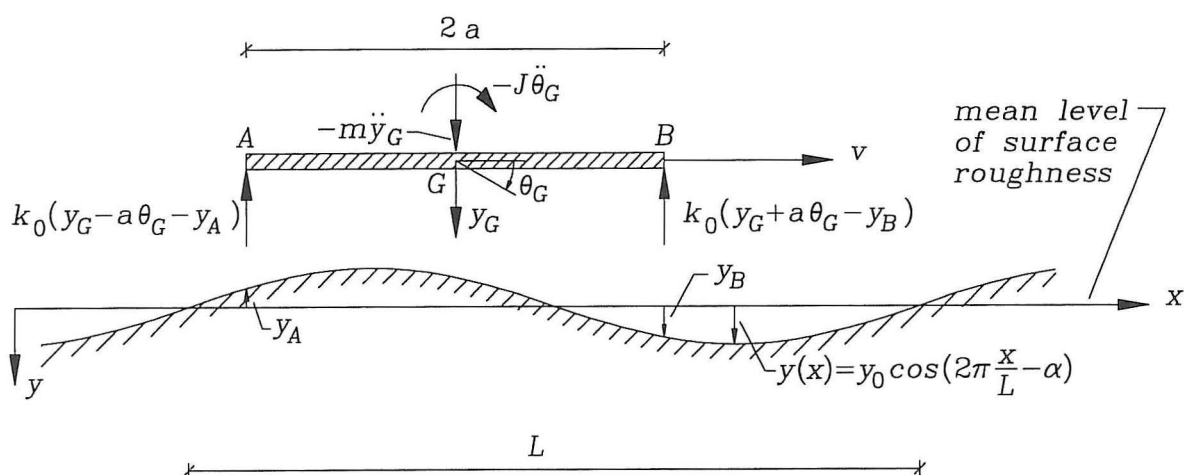


Fig. 1: Forces on a free vehicle beam.

The static equilibrium state of the beam AB is defined as the horizontal level of the beam at the velocity $v = 0$, and with the profile elevation $y_A = y_B = 0$ at the support points A and B . The system has 2 degrees of freedom, which are selected as the vertical displacement $y_G(t)$ and the rotation $\theta_G(t)$ of the centre of gravity from the thus defined equilibrium state with signs as shown in fig. 1. The compression of the springs at nodes A and B then becomes $y_G(t) - a\theta_G(t) - y_A(t)$ and $y_G(t) + a\theta_G(t) - y_B(t)$, respectively, where $y_A(t)$ and $y_B(t)$ are the profile elevation at the points A and B at the time t . If $t = 0$ is selected at the instant of time, where point A is at the position $x = 0$, the

abscissas of points A and B at the time t are given as $x_A = vt$ and $x_B = vt + 2a$, respectively. Hence, the profile elevation at the points A and B at the time t becomes

$$y_A(t) = y_0 \cos(\omega t - \alpha) \quad (1)$$

$$y_B(t) = y_0 \cos\left(\omega t + 4\pi \frac{a}{L} - \alpha\right) \quad (2)$$

$$\omega = 2\pi \frac{v}{L} \quad (3)$$

ω as given by (3) may be considered as an artificial circular excitation frequency. The beam AB is cut free from the springs, and the spring forces $k_0(y_G - a\theta_G - y_A)$ and $k_0(y_G + a\theta_G - y_B)$ are applied as external forces with signs as shown in fig. 1. Besides, the inertial load $-m\ddot{y}_G$ and the inertial moment $-J\ddot{\theta}_G$ are applied as external loads according to d'Alembert's principle with signs as shown in fig. 1. Next, the equations of motion can be formulated, expressing the vertical force equilibrium and the moment equilibrium around point G

$$\left. \begin{aligned} -m\ddot{y}_G &= k_0(y_G - a\theta_G - y_A(t)) + k_0(y_G + a\theta_G - y_B(t)) \\ -J\ddot{\theta}_G &= -k_0(y_G - a\theta_G - y_A(t))a + k_0(y_G + a\theta_G - y_B(t))a \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \ddot{y}_G \\ \ddot{\theta}_G \end{bmatrix} + 2k_0 \begin{bmatrix} 1 & 0 \\ 0 & a^2 \end{bmatrix} \begin{bmatrix} y_G \\ \theta_G \end{bmatrix} = k_0 \begin{bmatrix} 1 & 1 \\ -a & a \end{bmatrix} \begin{bmatrix} y_A(t) \\ y_B(t) \end{bmatrix} \quad (4)$$

It follows from (4) that the selected degrees of freedom decouple the mass and stiff matrix. This means that y_G and θ_G are modal coordinates. The undamped circular frequencies become

$$\left. \begin{aligned} \omega_1 &= \sqrt{\frac{2k_0}{m}} \\ \omega_2 &= \sqrt{\frac{2k_0 a^2}{J}} \end{aligned} \right\} \quad (5)$$

If the circular frequency ω as given by (3) is equal to either ω_1 or ω_2 , $y_A(t)$ and $y_B(t)$ will cause resonance of the system. Hence, v must be selected, so this is avoided. From (3) it then follows that

$$v \neq \left\{ \begin{aligned} &\frac{L}{2\pi} \sqrt{\frac{2k_0}{m}} \\ &\frac{L}{2\pi} \sqrt{\frac{2k_0 a^2}{J}} \end{aligned} \right. \quad (6)$$

Question 2:

If $\frac{a}{L} = \frac{1}{4}$ then it follows from (1) and (2) that

$$y_B(t) = -y_A(t) = -y_0 \cos(\omega t - \alpha) \quad (7)$$

The 1st differential equation (4) then becomes

$$m\ddot{y}_G + 2k_0 y_G = 0 \quad (8)$$

Because the excitation is 0, the stationary solution of (8) becomes

$$y_G(t) \equiv 0 \quad (9)$$

A person placed at the centre of gravity G is then at rest at all times.

PROBLEM 2

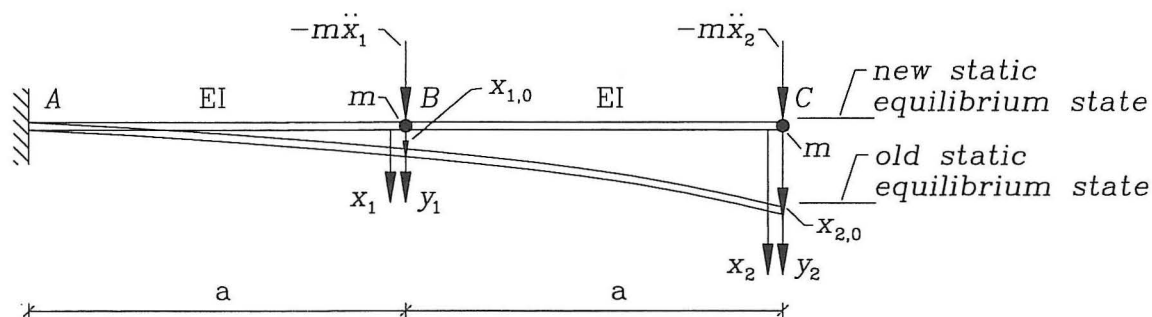


Fig. 1: Definition of static equilibrium states and related degrees of freedom.

Two static equilibrium states can be identified, which may be used as referential state for the motion of the system. The old static equilibrium state is the deflection of the beam under the influence of the gravity load $m_0 g$ at point C , whereas the new static equilibrium state is the horizontal rectilinear state, which the beam eventually will attain, when the eigenvibration following the release of the mass m_0 has been dissipated. The equations of motion and related initial values using both equilibrium states as referential states will be formulated below.

The beam is massless. Hence, the system has two degrees of freedom which are selected as the vertical displacements $x_1(t)$, $x_2(t)$ of the masses at the points B and C from the new static equilibrium state, or the displacements $y_1(t)$, $y_2(t)$ of the masses from the old static equilibrium states. The signs of the degrees of freedom have been defined in fig. 1. These are related as follows

$$\left. \begin{aligned} x_1(t) &= x_{1,0} + y_1(t) \\ x_2(t) &= x_{2,0} + y_2(t) \end{aligned} \right\} \quad (1)$$

$x_{1,0}$ and $x_{2,0}$ are the displacements of the masses from the new static equilibrium state, when the beam is statically loaded with the gravity load m_0g at point C .

Initially, the equations of motion are formulated with the new static equilibrium state as referential state. The inertial loads $-m\ddot{x}_1$ and $-m\ddot{x}_2$ are applied as external loads according to d'Alembert's principle. The equations of motion then read, cf. (3-1)

$$\left. \begin{aligned} x_1(t) &= \delta_{11}(-m\ddot{x}_1) + \delta_{12}(-m\ddot{x}_2) \\ x_2(t) &= \delta_{21}(-m\ddot{x}_1) + \delta_{22}(-m\ddot{x}_2) \end{aligned} \right\} \quad (2)$$

The flexibility matrix becomes, see (B-5)

$$\mathbf{D} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \frac{1}{6} \frac{a^3}{EI} \begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix} \quad (3)$$

The equations of motion then become

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \quad , \quad t > 0 \quad (4)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad , \quad \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad , \quad \mathbf{K} = \frac{6}{7} \frac{EI}{a^3} \begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix} \quad (5)$$

The initial displacement from the new static equilibrium state is caused by the gravity load m_0g at point C . Hence

$$\mathbf{x}_0 = \mathbf{x}(0) = \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} = \begin{bmatrix} \delta_{12} \\ \delta_{22} \end{bmatrix} m_0g = \frac{1}{6} \frac{a^3 m_0g}{EI} \begin{bmatrix} 5 \\ 16 \end{bmatrix} \quad (6)$$

The masses start at rest, when the string at point C is cut. Hence, the initial velocity is given as

$$\dot{\mathbf{x}}_0 = \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (7)$$

Next, the equations of motion relative to the old equilibrium state can be obtained by inserting (1) into (4), (6) and (7)

$$\left. \begin{aligned} \mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} &= -\mathbf{K}\mathbf{x}_0 \quad , \quad t > 0 \\ \mathbf{y}(0) &= \mathbf{0} \quad , \quad \dot{\mathbf{y}}(0) = \mathbf{0} \end{aligned} \right\} \quad (8)$$

where $\mathbf{y}^T(t) = [y_1(t), y_2(t)]$. In the following the initial value problem (4), (6) and (7) is solved. The circular eigenfrequencies ω_j and the eigenmodes $\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix}$ are obtained as non-trivial solutions to the homogeneous linear equations, cf. (3-42)

$$\begin{bmatrix} 16 - \lambda_j & -5 \\ -5 & 2 - \lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (9)$$

$$\lambda_j = \frac{7}{6} \frac{\omega_j^2 m a^3}{EI} \quad (10)$$

The characteristic equation becomes

$$(16 - \lambda_j)(2 - \lambda_j) - 5^2 = 0 \Rightarrow$$

$$\lambda_j = \begin{cases} 9 - \sqrt{74} & , \quad j = 1 \\ 9 + \sqrt{74} & , \quad j = 2 \end{cases} \quad (11)$$

$$\omega_j = \begin{cases} \sqrt{\frac{6}{7}(9 - \sqrt{74})} \sqrt{\frac{EI}{ma^3}} & , \quad j = 1 \\ \sqrt{\frac{6}{7}(9 + \sqrt{74})} \sqrt{\frac{EI}{ma^3}} & , \quad j = 2 \end{cases} \quad (12)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix} \quad , \quad j = 1, 2 \quad (13)$$

The first component $\Phi_1^{(j)}$ is determined from the first equation of (9) as follows

$$(16 - \lambda_j)\Phi_1^{(j)} - 5 \cdot 1 = 0 \Rightarrow$$

$$\Phi_1^{(j)} = \frac{5}{16 - \lambda_j} = \begin{cases} \frac{\sqrt{74}-7}{5} & , \quad j = 1 \\ -\frac{\sqrt{74}+7}{5} & , \quad j = 2 \end{cases} \quad (14)$$

The modal matrix and its inverse become, cf. (3-83)

$$\mathbf{P} = \begin{bmatrix} \frac{\sqrt{74}-7}{5} & -\frac{\sqrt{74}+7}{5} \\ 1 & 1 \end{bmatrix} \Rightarrow$$

$$\mathbf{P}^{-1} = \frac{5\sqrt{74}}{148} \begin{bmatrix} 1 & \frac{\sqrt{74}+7}{5} \\ -1 & \frac{\sqrt{74}-7}{5} \end{bmatrix} \quad (15)$$

The eigenvibration $\mathbf{x}(t)$ due to the initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(0) = \dot{\mathbf{x}}_0$ is given as, cf. (3-79)

$$\mathbf{x}(t) = a_1 \Phi^{(1)} \cos(\omega_1 t) + a_2 \Phi^{(2)} \cos(\omega_2 t) + b_1 \Phi^{(1)} \sin(\omega_1 t) + b_2 \Phi^{(2)} \sin(\omega_2 t) \quad (16)$$

By using (6), (7) and (15) the expansion coefficients a_1 , a_2 , b_1 , b_2 are given as, cf. (3-81), (3-82)

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{P}^{-1} \mathbf{x}_0 = \frac{1}{888} \frac{a^3 m_0 g}{EI} \begin{bmatrix} 1184 + 137\sqrt{74} \\ 1184 - 137\sqrt{74} \end{bmatrix} \quad (17)$$

$$\begin{aligned} \begin{bmatrix} \omega_1 b_1 \\ \omega_2 b_2 \end{bmatrix} &= \mathbf{P}^{-1} \dot{\mathbf{x}}_0 = \mathbf{0} \Rightarrow \\ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \quad (18)$$

From (15), (16) and (17) it follows that

$$\mathbf{x}(t) = \frac{1}{888} \frac{a^3 m_0 g}{EI} \left(\begin{bmatrix} 370 + 45\sqrt{74} \\ 1184 + 137\sqrt{74} \end{bmatrix} \cos(\omega_1 t) + \begin{bmatrix} 370 - 45\sqrt{74} \\ 1184 - 137\sqrt{74} \end{bmatrix} \cos(\omega_2 t) \right) \quad (19)$$

where ω_1 and ω_2 are given by (12).

Question 2:

As possible damping models the following Rayleigh model and 2 term Caughey models are considered, cf. (3-284), (3-289)

$$\mathbf{C} = a_0 \mathbf{M} + a_1 \mathbf{K} \quad (20)$$

$$\mathbf{C} = a_1 \mathbf{K} + a_2 \mathbf{K} \mathbf{M}^{-1} \mathbf{K} \quad (21)$$

$$\mathbf{C} = a_{-1} \mathbf{M} \mathbf{K}^{-1} \mathbf{M} + a_0 \mathbf{M} \quad (22)$$

where \mathbf{M} and \mathbf{K} are given by (5). The expansion coefficients of (20), (21) and (22) are obtained from the linear equations, cf. (3-290)

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\omega_1} & \frac{\omega_1}{2} \\ \frac{1}{2\omega_2} & \frac{\omega_2}{2} \end{bmatrix}^{-1} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \frac{2\omega_1\omega_2}{\omega_2^2 - \omega_1^2} \begin{bmatrix} \omega_2 & -\omega_1 \\ -\frac{1}{\omega_2} & \frac{1}{\omega_1} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \quad (23)$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{\omega_1}{2} & \frac{\omega_1^3}{2} \\ \frac{\omega_2}{2} & \frac{\omega_2^3}{2} \end{bmatrix}^{-1} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \frac{2}{\omega_1\omega_2(\omega_2^2 - \omega_1^2)} \begin{bmatrix} \omega_2^3 & -\omega_1^3 \\ -\omega_2 & \omega_1 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \quad (24)$$

$$\begin{bmatrix} a_{-1} \\ a_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\omega_1^3} & \frac{1}{2\omega_1} \\ \frac{1}{2\omega_2^3} & \frac{1}{2\omega_2} \end{bmatrix}^{-1} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \frac{2\omega_1^3\omega_2^3}{\omega_2^2 - \omega_1^2} \begin{bmatrix} \frac{1}{\omega_2} & -\frac{1}{\omega_1} \\ -\frac{1}{\omega_2^3} & \frac{1}{\omega_1^3} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \quad (25)$$

Inserting (12) into (23) a_0, a_1 can be determined. Inserting this result and (5) into (20) it then provides the following solution for the Rayleigh damping model

$$\mathbf{C} = \sqrt{\frac{EI m}{a^3}} \left(\begin{bmatrix} 0.10875 & 0.33935 \\ 0.33935 & 1.05892 \end{bmatrix} \zeta_1 + \begin{bmatrix} 7.04507 & -2.25770 \\ -2.25770 & 0.72351 \end{bmatrix} \zeta_2 \right) \quad (26)$$

The same procedure can next be performed for the Caughey models (21), (22), with the expansion coefficients determined by (24) and (25), respectively. Both models give exactly the solution (26) in accordance with the statement given subsequent to (3-294) and example 3-16.

PROBLEM 3

Question 1:

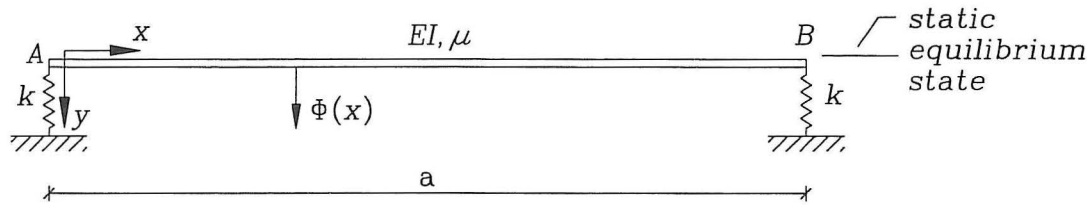


Fig. 1: Beam with a constant bending stiffness and a mass per unit length.

A local (x, y) coordinate system is defined for the beam element as shown in fig. 1. The beam has constant bending stiffness EI , constant mass per unit length μ , and the normal force in the static equilibrium state is $N = 0$. Hence, the eigenmode is given by (4-18), (4-19)

$$\Phi(x) = A \sin\left(\lambda \frac{x}{a}\right) + B \cos\left(\lambda \frac{x}{a}\right) + C \sinh\left(\lambda \frac{x}{a}\right) + D \cosh\left(\lambda \frac{x}{a}\right) \quad (1)$$

$$\lambda^4 = \frac{\mu \omega^2 a^4}{EI} \quad (2)$$

The boundary conditions at point $A (x = 0)$ and point $B (x = a)$ become, see (4-13)

$$\frac{d^2}{dx^2} \Phi(0) = 0 \quad (3)$$

$$EI \frac{d^3}{dx^3} \Phi(0) = -k \Phi(0) \quad (4)$$

$$\frac{d^2}{dx^2}\Phi(a) = 0 \quad (5)$$

$$EI \frac{d^3}{dx^3}\Phi(a) = k\Phi(a) \quad (6)$$

All boundary conditions are mechanical. (3) and (5) state that the bending moment is 0 at the end sections. (4) and (6) state that the shear forces at the end sections must balance the forces in the springs.

(3) and (4) imply

$$\frac{\lambda^2}{a^2}(-B + D) = 0 \quad (7)$$

$$\frac{\lambda^3}{a^3}(-A + C) = -\frac{\kappa}{a^3}(B + D) \quad (8)$$

where

$$\kappa = \frac{ka^3}{EI} \quad (9)$$

(7) implies $D = B$. (8) then implies $C = A - \frac{2\kappa}{\lambda^3}B$. Using these results, (1) is reduced to

$$\Phi(x) = A \left(\sin\left(\lambda \frac{x}{a}\right) + \sinh\left(\lambda \frac{x}{a}\right) \right) + B \left(\cos\left(\lambda \frac{x}{a}\right) + \cosh\left(\lambda \frac{x}{a}\right) - \frac{2\kappa}{\lambda^3} \sinh\left(\lambda \frac{x}{a}\right) \right) \quad (10)$$

Inserting (10) into (5) and (6) the following linear homogeneous equations are obtained for the determination of A and B

$$\left. \begin{aligned} \frac{\lambda^2}{a^2}(-\sin \lambda + \sinh \lambda)A + \frac{\lambda^2}{a^2}\left(-\cos \lambda + \cosh \lambda - \frac{2\kappa}{\lambda^3} \sinh \lambda\right)B &= 0 \\ \frac{\lambda^3}{a^3}(-\cos \lambda + \cosh \lambda)A + \frac{\lambda^3}{a^3}\left(\sin \lambda + \sinh \lambda - \frac{2\kappa}{\lambda^3} \cosh \lambda\right)B &= \\ \kappa(\sin \lambda + \sinh \lambda)A + \kappa\left(\cos \lambda + \cosh \lambda - \frac{2\kappa}{\lambda^3} \sinh \lambda\right)B & \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} K_{11}(\lambda) & K_{12}(\lambda) \\ K_{21}(\lambda) & K_{22}(\lambda) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (11)$$

$$K_{11}(\lambda) = \lambda^3(-\sin \lambda + \sinh \lambda) \quad (11a)$$

$$K_{12}(\lambda) = \lambda^3(-\cos \lambda + \cosh \lambda) - 2\kappa \sinh \lambda \quad (11b)$$

$$K_{21}(\lambda) = \lambda^3(-\cos \lambda + \cosh \lambda) - \kappa(\sin \lambda + \sinh \lambda) \quad (11c)$$

$$K_{22}(\lambda) = \lambda^3(\sin \lambda + \sinh \lambda) - \kappa(\cos \lambda + 3 \cosh \lambda) + \frac{2\kappa^2}{\lambda^3} \sinh \lambda \quad (11d)$$

Non-trivial solutions $A \neq 0 \vee B \neq 0$ are obtained if the determinant of (11) is 0. This provides the frequency condition

$$K_{11}(\lambda)K_{22}(\lambda) - K_{12}(\lambda)K_{21}(\lambda) = 0 \quad \Rightarrow$$

$$\lambda^6(\cos \lambda \cosh \lambda - 1) + 2\kappa \lambda^3(\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda) - 2\kappa^2 \sin \lambda \sinh \lambda = 0 \quad (12)$$

At the evaluation of the determinant, standard trigonometric and hyperbolic identities have been applied.

For $\kappa = 3$ the 6 lowest solutions of (12) become

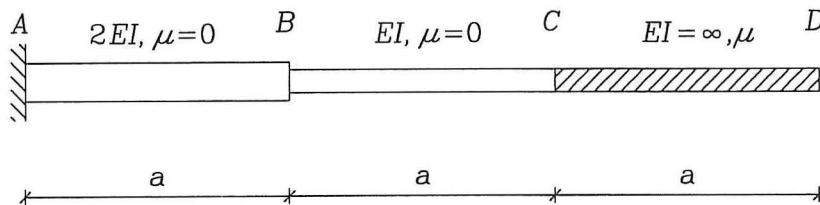
$$\lambda_j = \begin{cases} 1.54591 & , & j = 1 & \text{(symmetric)} \\ 2.05609 & , & j = 2 & \text{(anti-symmetric)} \\ 4.78626 & , & j = 3 & \text{(symmetric)} \\ 7.86561 & , & j = 4 & \text{(anti-symmetric)} \\ 11.00013 & , & j = 5 & \text{(symmetric)} \\ 14.13929 & , & j = 6 & \text{(anti-symmetric)} \end{cases} \quad (13)$$

Because the beam is symmetric around the midpoint the eigenvibrations separate into eigenvibrations with symmetric and anti-symmetric eigenmode. These turn out to change as indicated in (13). If the beam is infinitely stiff, the corresponding 2 eigenvalues are found as, see eq. (5) in problem 1, $\lambda_1 = \sqrt[4]{6} = 1.56508$ and $\lambda_2 = \sqrt[4]{18} = 2.05977$. From this comparison it is concluded that $\kappa = 3$ specifies very soft supporting springs.

2.10 September 4, 1991

Duration: 4h

PROBLEM 1



The figure shows a plane horizontal rectilinear beam structure $ABCD$, composed of the sub-beams AB , BC and CD . All sub-beams have the length a , and are considered infinitely stiff against axial elongations. The beams AB and BC are massless Bernoulli-Euler beams with the constant bending stiffness $2EI$ and EI , respectively. The beam CD is assumed to be infinitely stiff against bending deformations, and has the constant mass per unit length μ . The beam AB is fixed at point A , and the beam CD is free at point D . The sub-beams are fixed to each other at their end sections. Only small vertical vibrations of the structure from the static equilibrium state are considered.

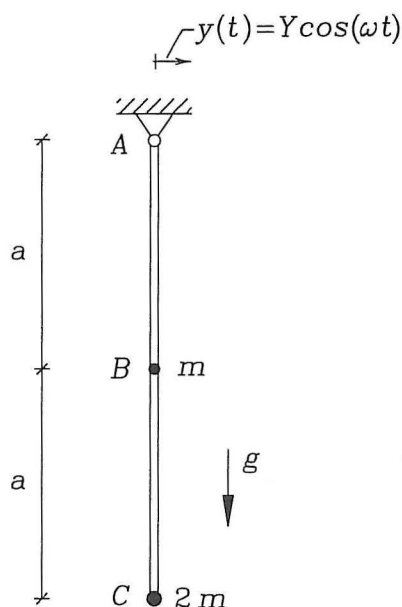
Question 1 (30%, $\mu = 18.8\%$)

Determine the undamped circular eigenfrequencies and undamped mode shapes of the structure.

Question 2 (10%, $\mu = 1.9\%$)

Specify a static inertial load on the structure, which is qualitatively in agreement with the inertial load in the first eigenvibration. Apply this load to estimate the first undamped circular eigenfrequency, using Rayleigh's fraction.

PROBLEM 2



The figure shows a vertical system of bars, consisting of the massless inextensible bars AB and BC , both of the length a . The structure is hinged at point A and the bars AB and BC are connected with a hinge at point B . At the points B and C concentrated masses of magnitudes m and $2m$ are applied, respectively. The movable support point A has the horizontal harmonic displacement from the static equilibrium state $y(t) = Y \cos(\omega t)$ with the amplitude Y and the circular frequency ω . This displacement and the vertical static equilibrium state of the bars define a plane. Only vibrations in this plane are considered, and the displacements of the masses from the vertical equilibrium state are assumed to be small compared to the length a . The acceleration of gravity is g .

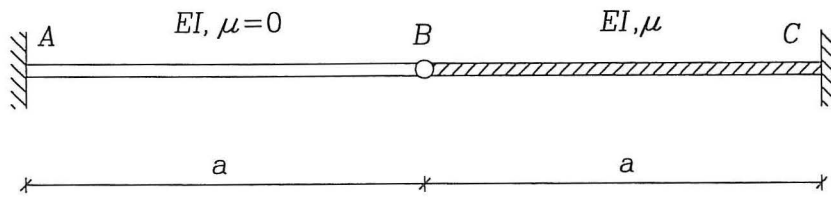
Question 1 (20%, $\mu = 2.4\%$)

Determine the stationary horizontal displacement of the points B and C , after the response from the initial conditions has been dissipated. Otherwise, the effect of any damping mechanism in the hinges or from the surroundings is ignored.

Question 2 (10%, $\mu = 0.0\%$)

Determine the values of the circular frequency ω , for which the stationary horizontal displacements of the points A and B are identical.

PROBLEM 3



The figure shows a plane horizontal rectilinear beam structure ABC , composed of the Bernoulli-Euler beams AB and BC . Both beams have the length a , the constant bending stiffness EI , and are infinitely stiff against axial elongations. The beam AB is massless and is fixed at point A . The beam BC has the constant mass per unit length μ , and is fixed at point C . The beams are joined by a hinge at point B . Only small vertical displacements in the plane of the structure from the static equilibrium state are considered. The influence on the vibrations from any normal forces in the static equilibrium state is ignored.

Question 1 (30%, $\mu = 14.2\%$)

Formulate a frequency condition for the determination of the undamped circular eigenfrequencies of the structure.

SOLUTIONS

PROBLEM 1

Question 1:

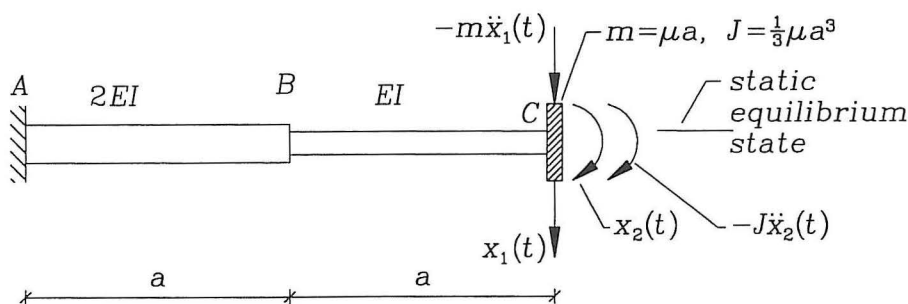


Fig. 1: Equivalent system with inertial loads.

The effect of the infinitely stiff beam CD is equivalent to a distributed mass applied to point C with the mass $m = \mu a$ and the mass moment of inertia $J = \frac{1}{3}\mu a^3$. Since the beams AB and BC are massless, the system has but 2 degrees of freedom, which are selected as the vertical displacement $x_1(t)$ and the rotation $x_2(t)$ of point C from the static equilibrium state with signs as defined in fig. 1. Next, the inertial force $-m\ddot{x}_1(t)$ and the inertial moment $-J\ddot{x}_2(t)$ are applied as external loads at point C according to d'Alembert's principle with signs as defined in fig. 1. The equations of motion then read, cf. (3-1)

$$\left. \begin{aligned} x_1(t) &= \delta_{11}(-\mu a \ddot{x}_1) + \delta_{12} \left(-\frac{1}{3} \mu a^3 \ddot{x}_2 \right) \\ x_2(t) &= \delta_{21}(-\mu a \ddot{x}_1) + \delta_{22} \left(-\frac{1}{3} \mu a^3 \ddot{x}_2 \right) \end{aligned} \right\} \quad (1)$$

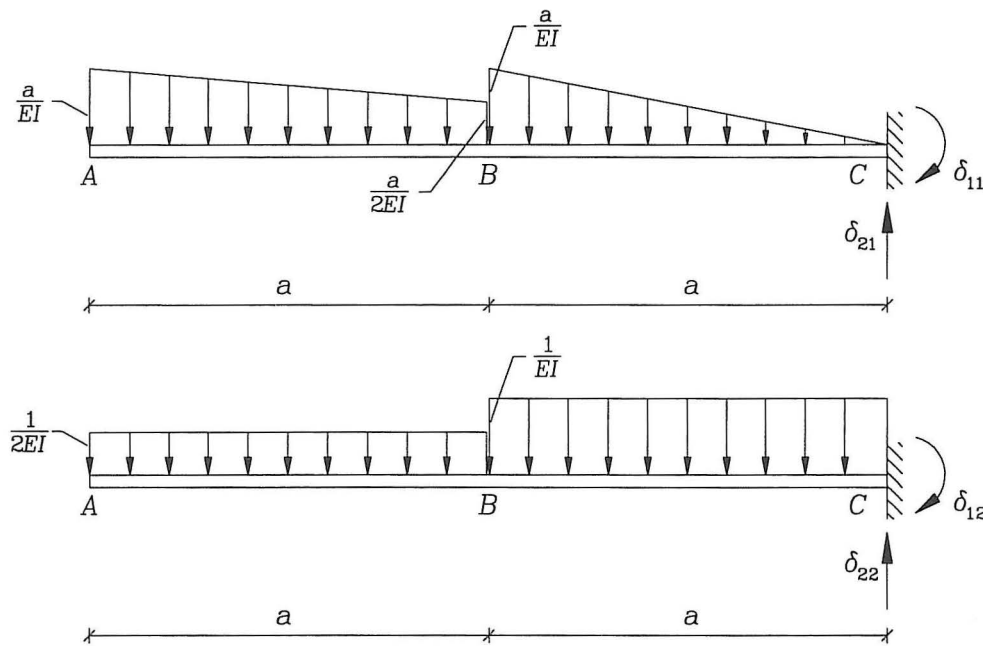


Fig. 2: Conjugated beam problems for the determination of flexibility coefficients δ_{11} , δ_{12} , δ_{21} , δ_{22} .

The flexibility coefficients δ_{11} , δ_{12} , δ_{21} , δ_{22} can be determined by any standard static analysis method. Below, the method of conjugated beams is applied. The conjugated beam of the present problem is a cantilever structure, fixed at point C and free at point A . The load per unit length of the conjugated beam is given by $M(x)/EI(x)$, where $M(x)$ is the moment field in the original beam from a unit load in the direction x_1 (at the determination of δ_{11} and δ_{21}), or from a unit moment in the direction of x_2 (at the determination of δ_{12} and δ_{22}). The coefficients of influence are then determined as the reaction moments and reaction forces at the support point C of the conjugated beam,

as shown in fig. 2. It then follows that

$$\left. \begin{aligned} \delta_{11} &= a \cdot \frac{a}{2EI} \cdot \frac{3}{2}a + \frac{a}{2} \cdot \frac{a}{2EI} \cdot \frac{5}{3}a + \frac{a}{2} \cdot \frac{a}{EI} \cdot \frac{2}{3}a = \frac{6}{4} \frac{a^3}{EI} \\ \delta_{21} &= a \cdot \frac{a}{2EI} + \frac{a}{2} \cdot \frac{a}{2EI} + \frac{a}{2} \cdot \frac{a}{EI} = \frac{5}{4} \frac{a^2}{EI} \\ \delta_{12} &= a \cdot \frac{1}{2EI} \cdot \frac{3}{2}a + a \cdot \frac{1}{EI} \cdot \frac{a}{2} = \frac{5}{4} \frac{a^2}{EI} \\ \delta_{22} &= a \cdot \frac{1}{2EI} + a \cdot \frac{1}{EI} = \frac{6}{4} \frac{a}{EI} \end{aligned} \right\} \quad (2)$$

Multiplication of (1) with the inverse flexibility matrix provides the following form of the equations of motion

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \quad (3)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ ax_2(t) \end{bmatrix}, \quad \mathbf{M} = \frac{11}{4} \frac{\mu a^4}{EI} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 6 & -5 \\ -5 & 6 \end{bmatrix} \quad (4)$$

The undamped circular eigenfrequencies ω_j and the eigenmodes $\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix}$ are then obtained as non-trivial solutions of the homogeneous linear equations, cf. (3-42)

$$\begin{bmatrix} 6 - \lambda_j & -5 \\ -5 & 6 - \frac{1}{3}\lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (5)$$

$$\lambda_j = \frac{11}{4} \frac{\omega_j^2 \mu a^4}{EI} \quad (6)$$

The characteristic equation becomes

$$(6 - \lambda_j) \left(6 - \frac{1}{3}\lambda_j \right) - 25 = 0 \Rightarrow \lambda_j = \begin{cases} 12 - \sqrt{111} & , \quad j = 1 \\ 12 + \sqrt{111} & , \quad j = 2 \end{cases} \quad (7)$$

$$\omega_j = \begin{cases} \sqrt{\frac{4}{11}(12 - \sqrt{111})} \sqrt{\frac{EI}{\mu a^4}} \simeq 0.7297 \sqrt{\frac{EI}{\mu a^4}} & , \quad j = 1 \\ \sqrt{\frac{4}{11}(12 + \sqrt{111})} \sqrt{\frac{EI}{\mu a^4}} \simeq 2.8627 \sqrt{\frac{EI}{\mu a^4}} & , \quad j = 2 \end{cases} \quad (8)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix} \quad , \quad j = 1, 2 \quad (9)$$

The first component $\Phi_1^{(j)}$ is determined from the first equation of (5)

$$(6 - \lambda_j)\Phi_1^{(j)} - 5 \cdot 1 = 0 \Rightarrow$$

$$\Phi_1^{(j)} = \frac{5}{6 - \lambda_j} = \begin{cases} \frac{6 + \sqrt{111}}{15} & , \quad j = 1 \\ \frac{6 - \sqrt{111}}{15} & , \quad j = 2 \end{cases} \quad (10)$$

Question 2:

The static load is estimated as

$$\mathbf{f} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (11)$$

(11) specifies a unit force at point C in the vertical direction. The corresponding displacement \mathbf{x} follows from (4) and (3-306)

$$\mathbf{x} = \mathbf{K}^{-1}\mathbf{f} = \begin{bmatrix} 6 & -5 \\ -5 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 6 \\ 5 \end{bmatrix} \quad (12)$$

The Rayleigh fraction then becomes, see (4), (12) and (3-308)

$$\omega_R^2 = \frac{\mathbf{x}^T \mathbf{K} \mathbf{x}}{\mathbf{x}^T \mathbf{M} \mathbf{x}} = \frac{1}{\frac{11}{4} \frac{\mu a^4}{EI}} \cdot \frac{\begin{bmatrix} 6 \\ 5 \end{bmatrix}^T \begin{bmatrix} 6 & -5 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix}}{\begin{bmatrix} 6 \\ 5 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix}} = \frac{792}{1463} \frac{EI}{\mu a^4} \Rightarrow$$

$$\omega_R = \sqrt{\frac{792}{1463}} \sqrt{\frac{EI}{\mu a^4}} \simeq 0.7358 \sqrt{\frac{EI}{\mu a^4}} \quad (13)$$

(13) only deviates 0.83% from the exact result (8).

PROBLEM 2

Question 1:

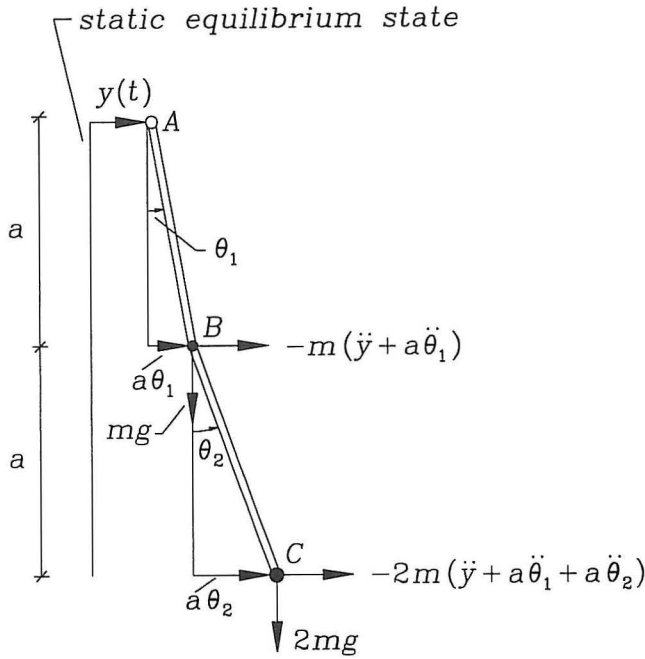


Fig 1: Load on a dynamically displaced system.

The bars are massless and inextensible. Hence, the system has but 2 degrees of freedom, which are selected as the rotational angles $\theta_1(t)$ and $\theta_2(t)$ of the bars from the vertical static equilibrium state with signs as defined in fig. 1. Assuming $|\theta_1(t)| \ll 1$ and $|\theta_2(t)| \ll 1$ the total horizontal displacements of the point masses, made up of the displacement $y(t)$ of point A and the contributions from the rotations of the bars, then become $y(t) + a\theta_1(t)$ and $y(t) + a\theta_1(t) + a\theta_2(t)$ for point B and C, respectively. Applying d'Alembert's principle, horizontal inertial loads of magnitude $-m(\ddot{y} + a\ddot{\theta}_1)$ and $-2m(\ddot{y} + a\ddot{\theta}_1 + a\ddot{\theta}_2)$ are next applied as external dynamic loads at point B and point C with signs as defined in fig. 1. Finally, the gravity loads mg and $2mg$ are acting on the masses. Throughout the motion these forces must balance each other so that the moment at the hinges in the points A and B are equal to zero. These equilibrium equations determine the linearized equations of motion of the system

$$\left. \begin{aligned} -m(\ddot{y} + a\ddot{\theta}_1) \cdot a - 2m(\ddot{y} + a\ddot{\theta}_1 + a\ddot{\theta}_2) \cdot 2a &= mg \cdot a\theta_1 + 2mg \cdot (a\theta_1 + a\theta_2) \\ -2m(\ddot{y} + a\ddot{\theta}_1 + a\ddot{\theta}_2) \cdot a &= 2mg \cdot a\theta_2 \end{aligned} \right\} \Rightarrow$$

$$\mathbf{M}\ddot{\Theta} + \mathbf{K}\Theta = -\mathbf{F}\ddot{y}(t) \quad (1)$$

$$\Theta(t) = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix}, \quad \mathbf{F} = \frac{1}{a} \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{K} = \omega_0^2 \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \quad (2)$$

$$\omega_0^2 = \frac{g}{a} \quad (3)$$

With $y(t) = Y \cos(\omega t)$ the acceleration of the support motion becomes $\ddot{y}(t) = -\omega^2 Y \cos(\omega t)$. The stationary response of the masses is then given as, cf. (3-100), (3-101) and (3-102)

$$\Theta(t) = \Theta_0 \cos(\omega t) \quad (4)$$

$$\Theta_0 = \mathbf{H}(\omega) \mathbf{F} \omega^2 Y \quad (5)$$

$$\begin{aligned} \mathbf{H}(\omega) &= (\mathbf{K} - \omega^2 \mathbf{M})^{-1} = \begin{bmatrix} 3\omega_0^2 - 5\omega^2 & 2\omega_0^2 - 4\omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 \end{bmatrix}^{-1} = \\ &= \frac{1}{\omega^4 - 6\omega_0^2\omega^2 + 3\omega_0^4} \begin{bmatrix} \omega_0^2 - \omega^2 & -2\omega_0^2 + 4\omega^2 \\ \omega^2 & 3\omega_0^2 - 5\omega^2 \end{bmatrix} \end{aligned} \quad (6)$$

From (2), (5) and (6) it follows that

$$\Theta_0 = \frac{\omega^2}{\omega^4 - 6\omega_0^2\omega^2 + 3\omega_0^4} \begin{bmatrix} 3\omega_0^2 - \omega^2 \\ 3\omega_0^2 \end{bmatrix} \frac{Y}{a} \quad (7)$$

$\Theta(t) = \Theta_0 \cos(\omega t)$ is in phase with the excitation $\ddot{y}(t) = -\omega^2 Y \cos(\omega t)$, and the amplitude Θ_0 is real, because the structural system is free of damping.

Question 2:

The points A and B have the same horizontal displacement if the bar AB remains vertical during the motion, i.e. if $\theta_1(t) \equiv 0$. From (7) it follows that this is the case in the stationary motion, if the circular excitation frequency is selected, so that

$$\begin{aligned} 3\omega_0^2 - \omega^2 &= 0 \Rightarrow \\ \omega &= \sqrt{3} \omega_0 = \sqrt{3} \sqrt{\frac{g}{a}} \end{aligned} \quad (8)$$

PROBLEM 3

Question 1:

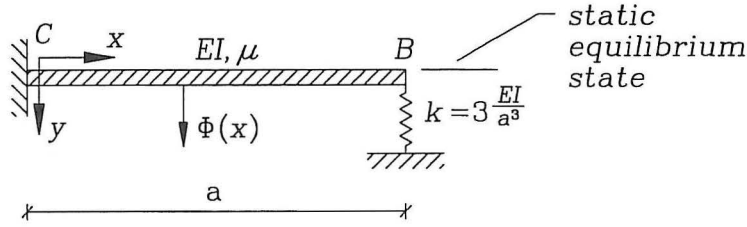


Fig. 1: Equivalent system.

The beam AB is massless. The influence of beam AB on the dynamic behaviour of the beam BC is then identical to that of a vertical linear elastic spring with the spring constant

$$k = 3 \frac{EI}{a^3} \quad (1)$$

The resulting equivalent system has been shown in fig. 1. As seen a local (x, y) -coordinate system has been introduced with origin at point C and the x -axis orientated against point B .

The beam in the equivalent system has constant bending stiffness EI , constant mass per unit length μ , and the normal force in the static equilibrium state $N = 0$. Hence, the eigenmode is given by (4-18) and (4-19)

$$\Phi(x) = A \sin\left(\lambda \frac{x}{a}\right) + B \cos\left(\lambda \frac{x}{a}\right) + C \sinh\left(\lambda \frac{x}{a}\right) + D \cosh\left(\lambda \frac{x}{a}\right) \quad (2)$$

$$\lambda^4 = \frac{\mu \omega^2 a^4}{EI} \quad (3)$$

The geometrical boundary at point C ($x = 0$) becomes, see (4-36)

$$\Phi(0) = \frac{d}{dx} \Phi(0) = 0 \quad (4)$$

Inserting (2) into (4) implies that $C = -A$ and $D = -B$. Then (2) reduces to

$$\Phi(x) = A \left(\sin\left(\lambda \frac{x}{a}\right) - \sinh\left(\lambda \frac{x}{a}\right) \right) + B \left(\cos\left(\lambda \frac{x}{a}\right) - \cosh\left(\lambda \frac{x}{a}\right) \right) \quad (5)$$

The mechanical boundary conditions at point $B(x = a)$ become, see (4-13)

$$\frac{d^2}{dx^2}\Phi(a) = 0 \quad (6)$$

$$\begin{aligned} EI \frac{d^3}{dx^3}\Phi(a) &= k\Phi(a) \Rightarrow \\ a^3 \frac{d^3}{dx^3}\Phi(a) &= \kappa\Phi(a) \end{aligned} \quad (7)$$

$$\kappa = \frac{ka^3}{EI} \quad (8)$$

By inserting (5) into (6) and (7) the following homogeneous equations are obtained for the determination of A and B

$$\left. \begin{aligned} \frac{\lambda^2}{a^2} \left(-A(\sin \lambda + \sinh \lambda) - B(\cos \lambda + \cosh \lambda) \right) &= 0 \\ \lambda^3 \left(-A(\cos \lambda + \cosh \lambda) - B(-\sin \lambda + \sinh \lambda) \right) &= \\ \kappa \left(A(\sin \lambda - \sinh \lambda) + B(\cos \lambda - \cosh \lambda) \right) & \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} \sin \lambda + \sinh \lambda & \cos \lambda + \cosh \lambda \\ \lambda^3(\cos \lambda + \cosh \lambda) + \kappa(\sin \lambda - \sinh \lambda) & \lambda^3(-\sin \lambda + \sinh \lambda) + \kappa(\cos \lambda - \cosh \lambda) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (9)$$

The frequency condition then becomes

$$\begin{aligned} &\lambda^3(-\sin^2 \lambda + \sinh^2 \lambda - \cos^2 \lambda - 2 \cos \lambda \cosh \lambda - \cosh^2 \lambda) + \\ &\kappa \left((\sin \lambda + \sinh \lambda)(\cos \lambda - \cosh \lambda) - (\sin \lambda - \sinh \lambda)(\cos \lambda + \cosh \lambda) \right) = 0 \Rightarrow \\ &\lambda^3(1 + \cos \lambda \cosh \lambda) + \kappa(\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda) = 0 \end{aligned} \quad (10)$$

(10) might have been obtained immediately by setting $m = 0$ in eq. (14) of problem 3, September 5, 1990 (compare fig. 1c of problem 3, September 5, 1990 with fig. 1 of the present problem).

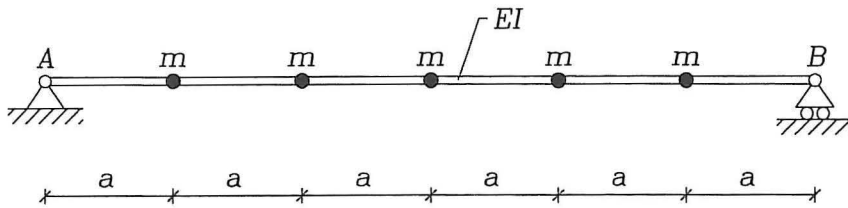
For $k = 3 \frac{EI}{a^3} \Rightarrow \kappa = 3$, the 6 lowest solutions of (10) become

$$\lambda_j = \begin{cases} 2.21350 & , \quad j = 1 \\ 4.72340 & , \quad j = 2 \\ 7.86097 & , \quad j = 3 \\ 10.99780 & , \quad j = 4 \\ 14.13823 & , \quad j = 5 \\ 17.27934 & , \quad j = 6 \end{cases} \quad (11)$$

2.11 June 12, 1992

Duration: 4h

PROBLEM 1

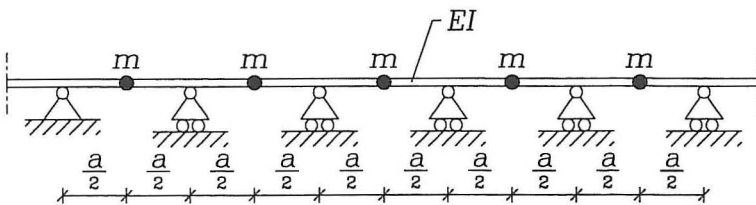


AB is a horizontal plane massless Bernoulli-Euler beam with the constant bending stiffness EI . The beam is simply supported at both end points A and B , and the length is $6a$. The beam is assumed to be infinitely stiff against axial deformations, and the contribution of the shear forces on the displacements is considered insignificant. At the 6th points of the beam 5 point masses of the magnitude m are applied. Only small vertical vibrations from the static equilibrium state of the beam are considered.

Question 1 (25% , $\mu = 17.3\%$)

Determine an estimation of the lowest undamped circular eigenfrequency of the structure by means of Rayleigh's fraction.

PROBLEM 2



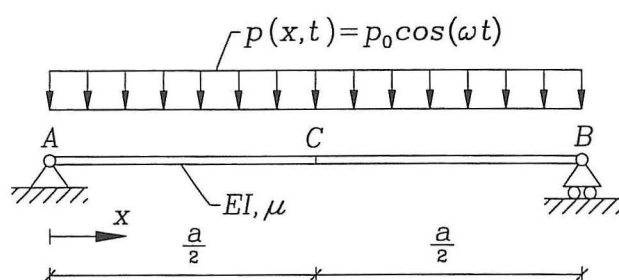
The figure shows an infinitely long horizontal plane Bernoulli-Euler beam with the constant bending stiffness EI . The beam extends continuously over an infinite number of intermediate simple supports. All spans have the width a , and in the middle of each span a point mass of magnitude m is applied. The beam is considered infinitely stiff against axial deformations, and the contribution of the shear forces on the displacements

is considered insignificant. Only small vertical vibrations from the static equilibrium state are considered, and the influence from any axial forces in the static equilibrium state on the dynamic behaviour is ignored.

Question 1 (20%, $\mu = 11.4\%$)

Make a sketch of the lowest eigenmode, and based on this determine the lowest undamped circular eigenfrequency.

PROBLEM 3



The beam AB is a horizontal plane Bernoulli-Euler beam with the constant bending stiffness EI and the constant mass per unit length μ . The beam is simply supported at both end sections A and B . Only small vertical vibrations from the static equilibrium state are considered.

The beam is loaded dynamically by a vertical harmonic varying load per unit length $p(x, t) = p_0 \cos(\omega t)$. ω is the circular excitation frequency and p_0 is the amplitude, which is constant along the beam.

Question 1 (10%, $\mu = 7.8\%$)

Determine the undamped circular eigenfrequencies, eigenmodes and modal masses of the beam.

Question 2 (15%, $\mu = 5.7\%$)

Determine the stationary dynamic bending moment at the midpoint C from the external dynamic load $p(x, t)$, when this has been acting for such a period of time that the response from the initial conditions has been dissipated. Otherwise, the effect of any damping mechanism on the system is ignored.

PROBLEM 1

Question 1:

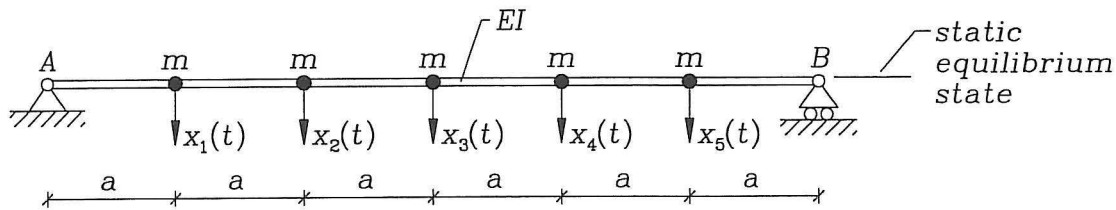


Fig. 1: The definition of degrees of freedom.

The beam is massless. Then the system has 5 degrees of freedom, which are selected as the vertical displacements $x_1(t), \dots, x_5(t)$ of the point masses from the static equilibrium state with signs as defined in fig. 1.

The flexibility coefficients become, see (B-1)

$$\delta_{ij} = \frac{1}{36} i(6-j)(12j-i^2-j^2) \frac{a^3}{EI} \quad , \quad 1 \leq i \leq j \leq 5 \quad (1)$$

The flexibility matrix and the mass matrix then become

$$\mathbf{D} = \frac{1}{18} \frac{a^3}{EI} \begin{bmatrix} 25 & 38 & 39 & 31 & 17 \\ 38 & 64 & 69 & 56 & 31 \\ 39 & 69 & 81 & 69 & 39 \\ 31 & 56 & 69 & 64 & 38 \\ 17 & 31 & 39 & 38 & 25 \end{bmatrix} \quad , \quad \mathbf{M} = m \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (2)$$

The static inertial load is estimated as

$$\mathbf{f} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad (3)$$

The static displacement vector \mathbf{x} caused by this load follows from (2) and (3), cf. (3-306)

$$\mathbf{x} = \mathbf{D}\mathbf{f} = \frac{1}{6} \frac{a^3}{EI} \begin{bmatrix} 50 \\ 86 \\ 99 \\ 86 \\ 50 \end{bmatrix} \quad (4)$$

The Rayleigh fraction then becomes, see (3-308)

$$\begin{aligned} \omega_R^2 &= \frac{\mathbf{x}^T \mathbf{f}}{\mathbf{x}^T \mathbf{M} \mathbf{x}} = \frac{2226}{29593} \frac{EI}{ma^3} \Rightarrow \\ \omega_1 &\simeq \omega_R = \sqrt{\frac{2226}{29593}} \sqrt{\frac{EI}{ma^3}} \simeq 0.27426 \sqrt{\frac{EI}{ma^3}} \end{aligned} \quad (5)$$

Note:

The eigenvalue problem for the 1st eigenmode reads, cf. (3-42)

$$\Phi^{(1)} = \mathbf{D}(\omega_1^2 \mathbf{M} \Phi^{(1)}) \quad (6)$$

where $\Phi^{(1)}$ is the 1st eigenmode. Hence, if $\mathbf{f} = k \cdot \omega_1^2 \mathbf{M} \Phi^{(1)}$, the displacement vector becomes $\mathbf{x} = k \cdot \Phi^{(1)}$, where k is an arbitrary constant. In this case Rayleigh's fraction determines the exact result ω_1^2 according to (3-53). \mathbf{x} given by (4) is merely an approximation to the 1st eigenmode. Using this approximation a new static load vector better than (3) can be calculated as follows

$$\mathbf{f} = \omega_1^2 \mathbf{M} \mathbf{x} = k \begin{bmatrix} 50 \\ 86 \\ 99 \\ 86 \\ 50 \end{bmatrix} \quad (7)$$

where $k = 1$ can be applied without any restrictions. The corresponding displacements become

$$\mathbf{x} = \mathbf{D}\mathbf{f} = \frac{1}{6} \frac{a^3}{EI} \begin{bmatrix} 3965 \\ 6867 \\ 7929 \\ 6867 \\ 3965 \end{bmatrix} \quad (8)$$

In this case Rayleigh's fraction becomes

$$\omega_R^2 = \frac{\mathbf{x}^T \mathbf{f}}{\mathbf{x}^T \mathbf{M} \mathbf{x}} = 7.5152976 \cdot 10^{-2} \frac{EI}{ma^3} \Rightarrow$$

$$\omega_1 \simeq \omega_R = 0.2741404 \sqrt{\frac{EI}{ma^3}} \quad (9)$$

(9) is very close to the exact result. (8) indicates a new and improved approximation to the 1st eigenvector. Next, based on this approximation a new static load vector $\mathbf{f} = \mathbf{M}\mathbf{x}$ can be calculated, from which an improved solution to the first circular eigenfrequency and eigenvector can be obtained, etc. The indicated iteration method, which is not contained in the lessons of the course, is termed inverse iteration (other names are Maxwell-Stodola iteration, Vianello's method, the power method etc.).

Finally, the theoretical solution for an equivalent beam with the smoothed out constant mass per unit length $\mu = \frac{m}{a}$ is stated, cf. (4-33)

$$\omega_1 = \frac{\pi^2}{(6a)^2} \sqrt{\frac{EI}{m/a}} = 0.2741557 \sqrt{\frac{EI}{ma^3}} \quad (10)$$

Obviously, this is an excellent approximation to the present problem.

PROBLEM 2

Question

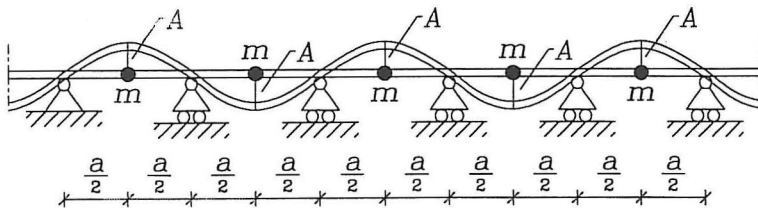


Fig. 1: The 1st eigenmode of the system.

The lowest eigenmode must have the shape shown in fig. 1. At the supports, the displacement is zero, and the curvature changes the sign due to the symmetric. Then the bending moment in the first eigenmode must be zero at the supports.

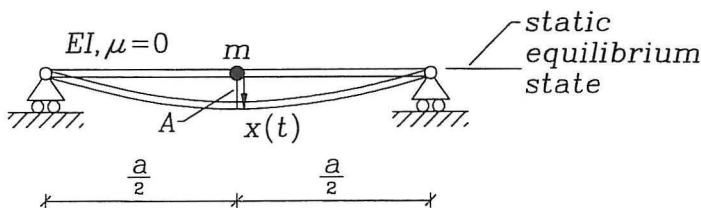


Fig. 2: The equivalent system for the analyses of the 1st eigenvibration.

Eigenvibrations in the lowest eigenmode can then be analysed by means of the equivalent system shown in fig. 2 with the fixed simple supports at the end sections. The system has a single degree of freedom $x(t)$, which is selected as the vertical displacement from the static equilibrium state of the point mass m . The equation of motion reads

$$x(t) = \delta_{11}(-m\ddot{x}) \quad , \quad \delta_{11} = \frac{1}{48} \frac{a^3}{EI} \Rightarrow$$

$$\omega_1 = \sqrt{\frac{1}{m\delta_{11}}} = \sqrt{48} \sqrt{\frac{EI}{ma^3}} \quad (1)$$

Note:

Similarly, higher order eigenvibrations can be analysed considering equivalent systems of increasing complexity. Below, in table 1 some eigenmodes have been indicated, which can immediately be sketched. The 1st, 2nd, 3rd, ... eigenmodes turn out to be periodic with a period P of 2 spans, 4 spans, 6 spans, ..., respectively. Besides, within each period the eigenmodes are anti-symmetric around the midpoint. The indicated ordinates of the eigenmodes can be determined from the orthogonality property, see (3-150)

$$\Phi^{(i)T} \mathbf{M} \Phi^{(j)} = 0 \quad , \quad i \neq j \quad \Rightarrow$$

$$\Phi^{(i)T} \Phi^{(j)} = 0 \quad , \quad i \neq j \quad (2)$$

because $\mathbf{M} = m\mathbf{I}$. In this case $\Phi^{(i)}$ has infinite dimension. However, the orthogonality property can be analysed because of the periodicity of the eigenmodes. As seen from table 1, the lowest and the highest eigenmode are analysed by especially simple equivalent systems.

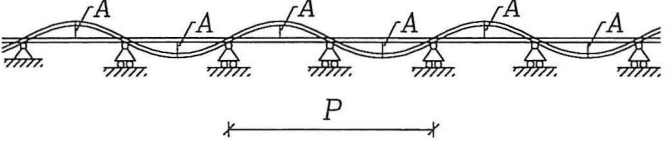
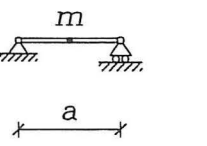
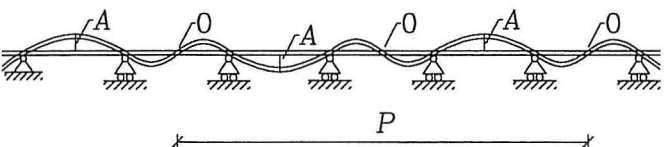
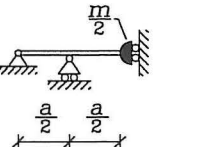
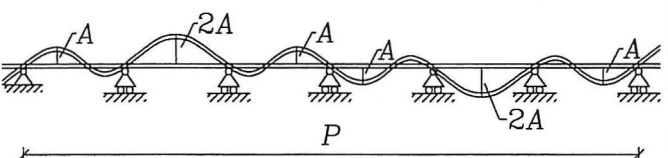
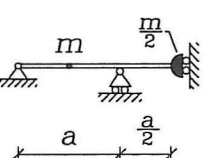
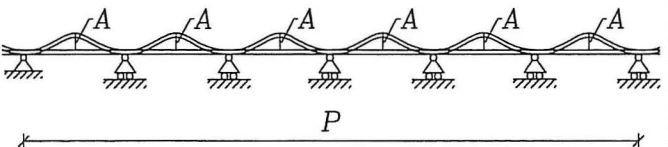
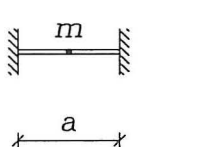
Eigen-vibr. i	$\frac{\omega_i}{\sqrt{\frac{EI}{ma^3}}}$	Mode shape	Equivalent system
1	$\sqrt{48}$ (=6.93)		
2	$\sqrt{\frac{768}{7}}$ (=10.47)		
3	$\sqrt{\frac{1920}{13}}$ (=12.15)		 Only highest eigenmode
...
∞	$\sqrt{192}$ (=13.86)		

Table 1: The eigenmodes of the infinite dimensional system.

PROBLEM 3

Question 1:

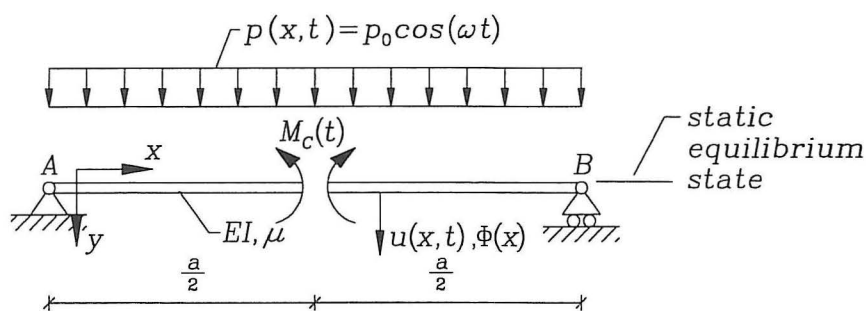


Fig. 1: A harmonically excited, simply supported beam. Definition of signs.

The undamped circular eigenfrequencies become, see (4-33)

$$\omega_j = j^2 \pi^2 \sqrt{\frac{EI}{\mu a^4}} \quad , \quad j = 1, 2, \dots \quad (1)$$

The eigenmodes become, see (4-31)

$$\Phi^{(j)}(x) = \sin \left(j \pi \frac{x}{a} \right) \quad , \quad j = 1, 2, \dots \quad (2)$$

With the normalization of the eigenmodes as follows from (2), the modal masses become, cf. (4-64) and (4-65)

$$M_j = \int_0^a \mu (\Phi^{(j)}(x))^2 dx = \frac{1}{2} \mu a \quad , \quad j = 1, 2, \dots \quad (3)$$

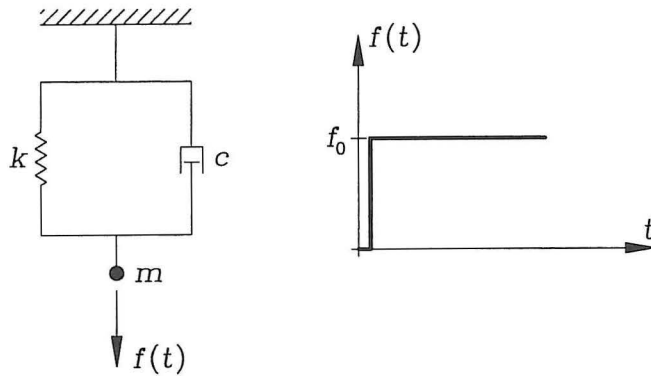
Question 2:

The solution of problem 2 has been included as example 4-6 in the textbook and will not be reiterated here.

2.12 June 16, 1993

Duration 4h

PROBLEM 1



The figure shows a system of a single degree of freedom consisting of a linear elastic spring with the spring constant k in parallel with a linear viscous damper with the damping constant c . Both are connected to the point mass m , which can only be moved in the vertical direction. The system is at rest at the time $t = 0$, where a step load $f(t)$ is applied to the mass in the vertical direction. The magnitude of the step load is f_0 as shown in the figure.

Question 1 (15% , $\mu = 5.6\%$)

Determine the displacement of the mass as a function of time.

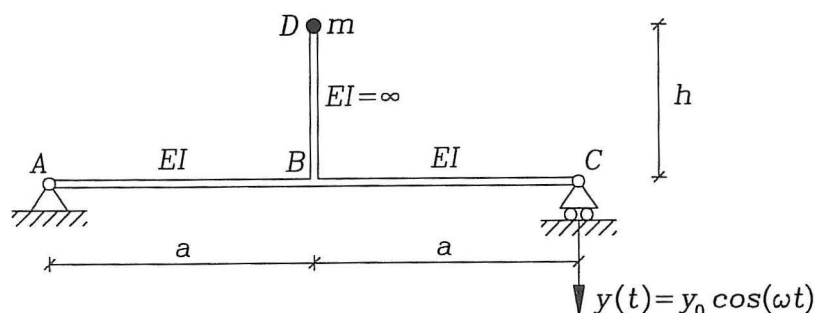
Question 2 (10% , $\mu = 1.6\%$)

Determine the numerically largest displacement of the mass.

The student may find the following indefinite integral useful at the solution of the problem

$$\int e^{ax} \sin(bx) dx = e^{ax} \frac{a \sin(bx) - b \cos(bx)}{a^2 + b^2}$$

PROBLEM 2



The figure shows a plane horizontal massless Bernoulli-Euler beam ABC with the length $2a$. The beam is rigidly simply supported at point A and movably simply supported at point C . The beam ABC has constant bending stiffness and is infinitely stiff against axial elongations. Vertical to the beam ABC at the midpoint B a plane massless beam BD of the length h is applied, which are infinitely stiff in bending and against axial elongations. The structure is excited by a vertical harmonic motion $y(t) = y_0 \cos(\omega t)$ at the support of point C . Only small vibrations from the static equilibrium state in the plane of the structure are considered.

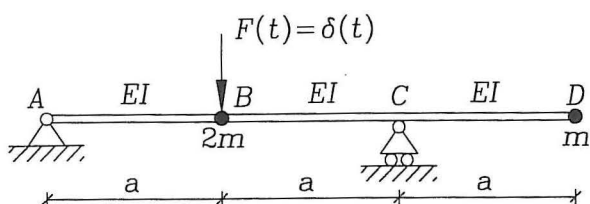
Question 1 (15%, $\mu = 4.0\%$)

Determine the value of h , for which the circular eigenfrequencies of the structure are identical.

Question 2 (10%, $\mu = 0.9\%$)

Determine the stationary, dynamic bending moment in the beam ABC immediately to the left of point B , when the excitation has been acting for such period of time that the moment response from the initial conditions has been dissipated. Otherwise, the effect of any damping mechanism acting on the system is ignored.

PROBLEM 3



The figure shows a plane horizontal massless Bernoulli-Euler beam $ABCD$ with the length $3a$. The beam is rigidly simply supported at point A , supported by a movable simple support at point C , and is free at point D . The sub-beams AB , BC , CD all have constant bending stiffness EI , and the same length a . At the points B and D point masses are applied of magnitudes $2m$ and m , respectively. The beam is assumed to be infinitely stiff against axial elongations, and the influence on the response from any damping mechanism is ignored. Only small vertical vibrations from the static equilibrium state are considered.

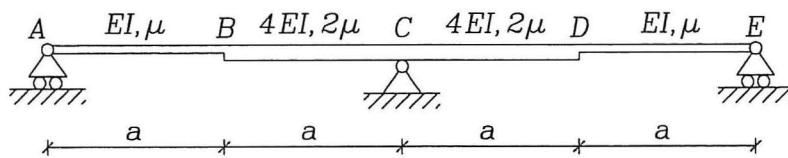
Question 1 (10%, $\mu = 8.2\%$)

Determine the undamped circular eigenfrequencies and the eigenmodes of the structure.

Question 2 (15%, $\mu = 4.3\%$)

Determine the motion of the masses from the static equilibrium state, if the system is at rest at the time $t = 0$, where a vertical unit impulse $F(t) = \delta(t)$ is applied at the mass at point B .

PROBLEM 4



The figure shows a continuous, horizontal plane Bernoulli-Euler beam $ABCDE$. All the sub-beams AB , BC , CD and DE have the length a , constant bending stiffness and constant mass per unit length.

The beams AB and DE both have the bending stiffness EI and the mass per unit length μ . The beams BC and CD both have the bending stiffness $4EI$ and the mass per unit length 2μ . The beam is rigidly supported by a rigid simple support at point C , and by movable simple supports at the end sections at the points A and E . The beam is assumed to be infinitely stiff against axial elongations, and the influence on the dynamic behaviour from any axial forces in the static equilibrium state is ignored. Further, only small vertical vibrations from the static equilibrium state in the plane of the structure are considered, and the influence from any damping mechanism is ignored.

Question 1 (10%, $\mu = 7.6\%$)

Prove that the eigenvibrations of the structure can be analysed by means of equivalent reduced systems, corresponding to symmetric and anti-symmetric eigenvibrations, respectively.

SOLUTIONS

PROBLEM 1

Question 1:

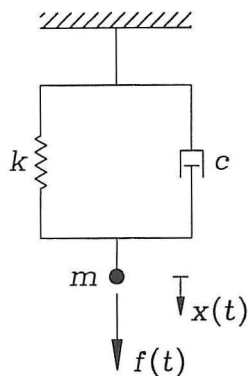


Fig. 1: Single degree-of-freedom system.

With the initial values $x(0) = \dot{x}(0) = 0$, the solution to the forced vibration problem becomes, cf. (2-110), (2-121)

$$\begin{aligned}
 x(t) &= \int_0^t h(t-\tau)f(\tau)d\tau = f_0 \int_0^t \frac{e^{-\zeta\omega_0(t-\tau)}}{m\omega_d} \sin(\omega_d(t-\tau))d\tau = \\
 &= \frac{f_0}{m\omega_d} \int_0^t e^{-\zeta\omega_0 u} \sin(\omega_d u)du = \\
 &= \frac{f_0}{m\omega_d} \left[e^{-\zeta\omega_0 u} \frac{-\zeta\omega_0 \sin(\omega_d u) - \omega_d \cos(\omega_d u)}{\zeta^2\omega_0^2 + \omega_d^2} \right]_0^t = \\
 &= \frac{f_0}{m\omega_0^2} \left[1 - e^{-\zeta\omega_0 t} \left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_d t) \right) \right]
 \end{aligned} \tag{1}$$

where

$$\omega_0^2 = \frac{k}{m} \tag{2}$$

$$\zeta = \frac{c}{2\sqrt{km}} \tag{3}$$

$$\omega_d = \omega_0 \sqrt{1-\zeta^2} \tag{4}$$

Question 2:

The maximum condition reads

$$\begin{aligned}\frac{dx(t)}{dt} &= f_0 h(t) = \frac{f_0}{m\omega_d} e^{-\zeta\omega_0 t} \sin(\omega_d t) = 0 \Rightarrow \\ \sin(\omega_d t) &= 0 \Rightarrow \\ t &= \frac{p\pi}{\omega_d}, \quad p = 1, 2, \dots\end{aligned}\quad (5)$$

Due to the damping the global maximum is then attained at the first local maximum, i.e. for $p = 1$, corresponding to $t = \pi/\omega_d$. Inserting $\omega_d t = \pi$ into (1), the maximum response is obtained as follows

$$\begin{aligned}|x(t)|_{\max} &= \frac{|f_0|}{m\omega_0^2} \left[1 - \exp\left(-\frac{\zeta\omega_0\pi}{\omega_d}\right) \left(\cos(\pi) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\pi) \right) \right] = \\ &= \frac{|f_0|}{k} \left(1 + \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi\right) \right)\end{aligned}\quad (6)$$

PROBLEM 2

Question 1:

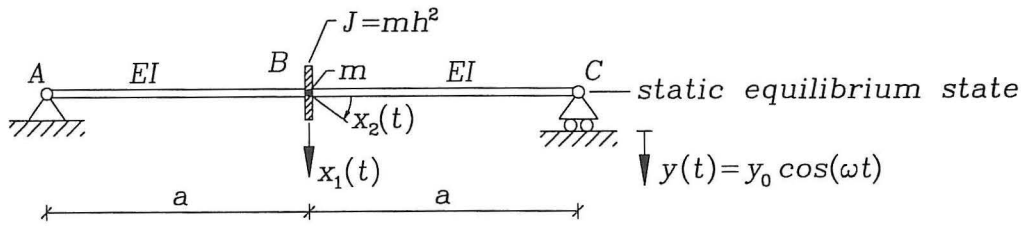


Fig. 1: Equivalent system.

The effect of the beam BD can be replaced by the distributed mass m with the moment of mass inertia $J = mh^2$ at point B as shown in fig. 1. The system has 2 degrees of freedom which are selected as the vertical displacement $x_1(t)$ and rotation $x_2(t)$ of point B from the static equilibrium state and with signs as defined in fig. 1.

Application of d'Alembert's principle provides

$$\left. \begin{aligned}x_1(t) &= \frac{1}{2}y(t) + \delta_{11}(-m\ddot{x}_1) + \delta_{12}(-J\ddot{x}_2) \\ x_2(t) &= \frac{1}{2a}y(t) + \delta_{12}(-m\ddot{x}_1) + \delta_{22}(-J\ddot{x}_2)\end{aligned} \right\} \quad (1)$$

The flexibility coefficients entering (1) can be found as

$$\left. \begin{aligned} \delta_{11} &= \frac{1}{6} \frac{a^3}{EI} \\ \delta_{12} &= 0 \\ \delta_{22} &= \frac{1}{6} \frac{a}{EI} \end{aligned} \right\} \quad (2)$$

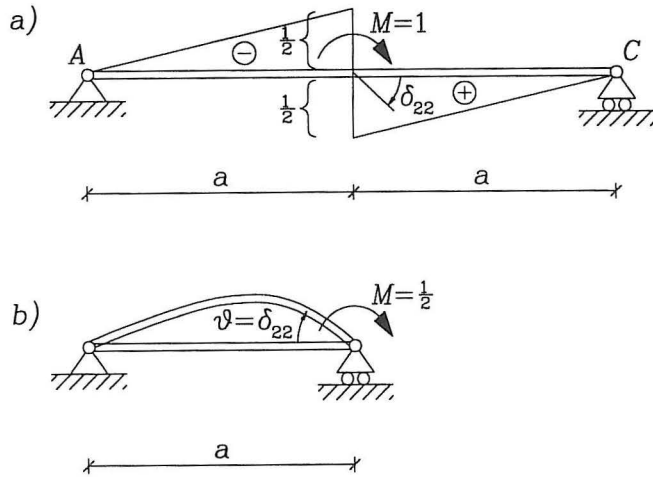


Fig 2: a) Load for calculation of δ_{22} . b) Equivalent, reduced system for calculation of δ_{22} .

δ_{22} is easily calculated from the equivalent system shown in fig. 2b, where the relationship between M and θ is known to be

$$M = 3 \frac{EI}{a} \theta \quad (3)$$

Inserting $M = \frac{1}{2}$, $\theta = \delta_{22} = \frac{1}{6} \frac{a}{EI}$ is obtained. By inserting (2) into (1) the following equations of motion are finally obtained

$$\left. \begin{aligned} m\ddot{x}_1 + \frac{6EI}{a^3} x_1 &= 3 \frac{EI}{a^3} y_0 \cos(\omega t) \\ m\ddot{x}_2 + \frac{6EI}{ah^2} x_3 &= 3 \frac{EI}{a^2 h^2} y_0 \cos(\omega t) \end{aligned} \right\} \quad (4)$$

As seen, the equations of motion are decoupled with the selected degrees of freedom. The circular eigenfrequencies become

$$\left. \begin{aligned} \omega_1 &= \sqrt{\frac{6EI}{ma^3}} \\ \omega_2 &= \sqrt{\frac{6EI}{mah^2}} \end{aligned} \right\} \quad (5)$$

Hence, $\omega_1 = \omega_2$ if $h = a$.

Note:

A similar problem has been included in the textbook as example 3-12.

Question 2:

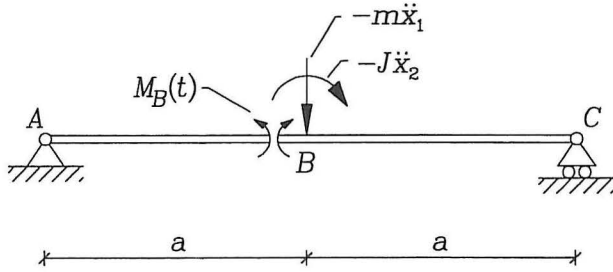


Fig. 3. Inertial loads on simply supported beam.

The bending moment $M_B(t)$ to the left of point B is made up of a sum of contributions from the quasi-static displacement of point C , and contributions from the inertial loadings at point B . Since the quasi-static displacement of the beam is a stiff-body mode, no bending moments are introduced. With the sign defined in fig. 3 one then has for $M_B(t)$

$$M_B(t) = \frac{1}{4}2a(-m\ddot{x}_1) - \frac{1}{2}(-J\ddot{x}_2) = -\frac{ma}{2}\ddot{x}_1 + \frac{mh^2}{2}\ddot{x}_2 \quad (6)$$

Using (5) the stationary solution of (4) is found to be

$$\left. \begin{aligned} x_1(t) &= \frac{1}{2} \frac{\omega_1^2}{\omega_1^2 - \omega^2} y_0 \cos(\omega t) \\ x_2(t) &= \frac{1}{2} \frac{\omega_2^2}{\omega_2^2 - \omega^2} \frac{y_0}{a} \cos(\omega t) \end{aligned} \right\} \Rightarrow$$

$$\left. \begin{aligned} \ddot{x}_1(t) &= -\frac{1}{2} \frac{\omega^2 \omega_1^2}{\omega_1^2 - \omega^2} y_0 \cos(\omega t) \\ \ddot{x}_2(t) &= -\frac{1}{2} \frac{\omega^2 \omega_2^2}{\omega_2^2 - \omega^2} \frac{y_0}{a} \cos(\omega t) \end{aligned} \right\} \quad (7)$$

Finally, from (6) and (7) it follows that

$$M_B(t) = \frac{ma}{4} \left(\frac{\omega_1^2}{\omega_1^2 - \omega^2} - \frac{\omega_2^2}{\omega_2^2 - \omega^2} \frac{h^2}{a^2} \right) \omega^2 y_0 \cos(\omega t) \quad (8)$$

PROBLEM 3

Question 1:

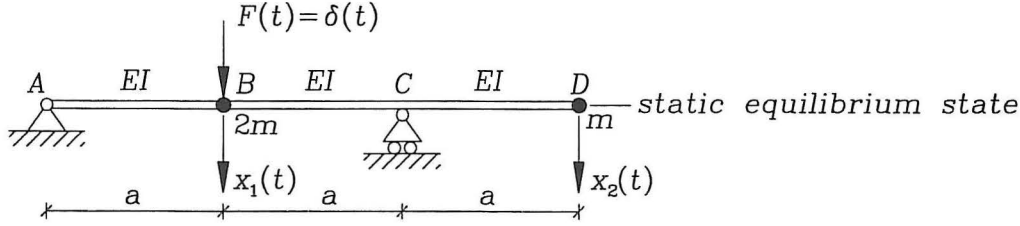


Fig. 1: 2 degrees-of-freedom system.

Since the beam is massless the system has but 2 degrees of freedom which are selected as the vertical displacements $x_1(t)$ and $x_2(t)$ of the masses at point B and point D from the static equilibrium state with signs as shown in fig. 1. Using d'Alembert's principle the equations of motion read

$$\left. \begin{aligned} x_1(t) &= \delta_{11}(-2m\ddot{x}_1 + F(t)) + \delta_{12}(-m\ddot{x}_2) \\ x_2(t) &= \delta_{12}(-2m\ddot{x}_1 + F(t)) + \delta_{22}(-m\ddot{x}_2) \end{aligned} \right\} \quad (1)$$

where the flexibility coefficients can be calculated as follows, cf. (B-1), (B-2), (B-3)

$$\left. \begin{aligned} \delta_{11} &= \frac{2}{12} \frac{a^3}{EI} \\ \delta_{12} &= -\frac{3}{12} \frac{a^3}{EI} \\ \delta_{22} &= \frac{12}{12} \frac{a^3}{EI} \end{aligned} \right\} \quad (2)$$

Inserting (2) into (1) and inverting the flexibility matrix the following equations of motion are obtained

$$\left. \begin{aligned} \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \frac{4}{5} \frac{EI}{a^3} \begin{bmatrix} 12 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} F(t) \quad , \quad t > 0^- \\ x_1(0^-) = x_2(0^-) = 0 \quad , \quad \dot{x}_1(0^-) = \dot{x}_2(0^-) = 0 \end{aligned} \right\} \quad (3)$$

Eigenvalues and eigenmodes are determined from the homogeneous linear equations

$$\begin{bmatrix} 12 - 2\lambda & 3 \\ 3 & 2 - \lambda \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (4)$$

$$\lambda = \frac{5}{4} \frac{m\omega^2 a^3}{EI} \quad (5)$$

The characteristic equation becomes

$$\det \left(\begin{bmatrix} 12 - 2\lambda & 3 \\ 3 & 2 - \lambda \end{bmatrix} \right) = 0 \Rightarrow$$

$$2\lambda^2 - 16\lambda + 15 = 0 \Rightarrow$$

$$\left. \begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix} \right\} = 4 \mp \sqrt{8.5} \quad (6)$$

$$\left. \begin{matrix} \omega_1 \\ \omega_2 \end{matrix} \right\} = \sqrt{\frac{4}{5}(4 \mp \sqrt{8.5})} \sqrt{\frac{EI}{ma^3}} \quad (7)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix}, \quad j = 1, 2 \quad (8)$$

The first component $\Phi_1^{(j)}$ is determined from the first equation of (4)

$$(12 - 2\lambda_j)\Phi_1^{(j)} + 3 \cdot 1 = 0 \Rightarrow$$

$$\Phi_1^{(j)} = \frac{3}{2\lambda_j - 12} = \begin{cases} \frac{4 - \sqrt{34}}{6}, & j = 1 \\ \frac{4 + \sqrt{34}}{6}, & j = 2 \end{cases} \quad (9)$$

The modal masses become

$$M_j = \Phi^{(j)T} \mathbf{M} \Phi^{(j)} = \begin{cases} \frac{1}{9}(34 - 4\sqrt{34})m, & j = 1 \\ \frac{1}{9}(34 + 4\sqrt{34})m, & j = 2 \end{cases} \quad (10)$$

Question 2:

Since $\mathbf{x}(0^-) = \mathbf{0}$ it follows from (3-143) that

$$\mathbf{x}(t) = \int_{0^-}^t \mathbf{h}(t - \tau) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \delta(\tau) d\tau = \mathbf{h}_1(t) \quad (11)$$

$\mathbf{h}_1(t)$ signifies the first column in $\mathbf{h}(t)$. From (3-190), (3-195) it follows that

$$\mathbf{h}(t) = \sum_{j=1}^2 h_j(t) \Phi^{(j)} \Phi^{(j)T} \quad (12)$$

$$h_j(t) = \frac{1}{M_j \omega_j} \sin(\omega_j t) \quad (13)$$

Inserting (8), (9), (10), (12), (13) into (11) provides the solution

$$\begin{aligned} \mathbf{x}(t) &= h_1(t) \begin{bmatrix} (\Phi_1^{(1)})^2 \\ \Phi_1^{(1)} \end{bmatrix} + h_2(t) \begin{bmatrix} (\Phi_1^{(2)})^2 \\ \Phi_1^{(2)} \end{bmatrix} = \\ &= \frac{1}{m\omega_1\sqrt{34}} \begin{bmatrix} \frac{\sqrt{34}-4}{4} \\ -\frac{3}{2} \end{bmatrix} \sin(\omega_1 t) + \frac{1}{m\omega_2\sqrt{34}} \begin{bmatrix} \frac{\sqrt{34}+4}{4} \\ \frac{3}{2} \end{bmatrix} \sin(\omega_2 t) \end{aligned} \quad (14)$$

$$\left(\text{Check: } x_1(0) = x_2(0) = 0, \dot{x}_1(0) = \frac{1}{2m}, \dot{x}_2(0) = 0 \right)$$

PROBLEM 4

Question 1:

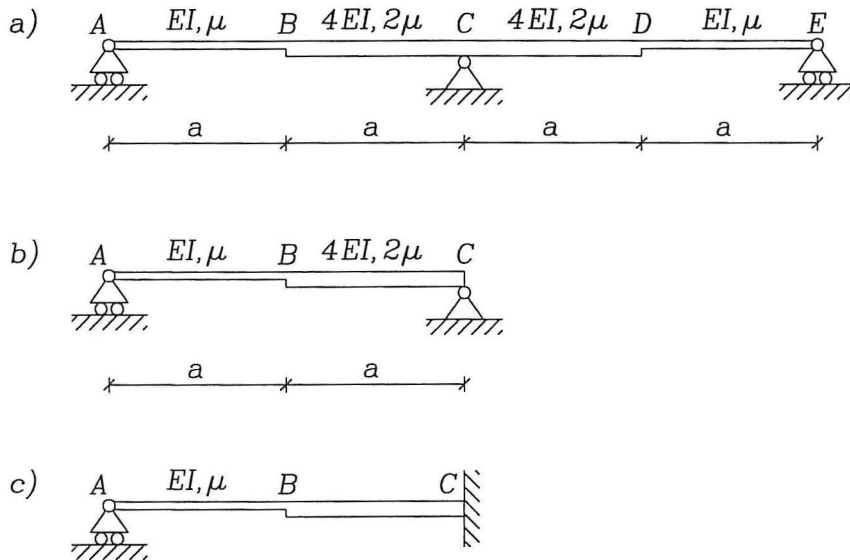


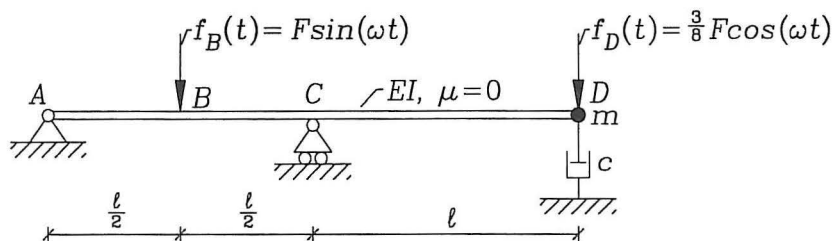
Fig. 1. a) Symmetric, continuous, homogeneous beam. b) Equivalent system for anti-symmetric eigenvibrations. c) Equivalent system for symmetric eigenvibrations.

The considered structure shown in fig. 1a is symmetric around point C . Hence, the eigenvibrations separate into anti-symmetric and symmetric eigenvibrations. The equivalent systems for the analysis of anti-symmetric eigenvibrations and symmetric eigenvibrations have been shown in fig. 1b and fig. 1c, respectively.

2.13 August 31, 1993

Duration 4h

PROBLEM 1



The figure shows a horizontal plane massless Bernoulli-Euler beam free of damping, and with the constant bending stiffness EI . The beam is rigidly, simply supported at point A , and supported by a movable simple support at point C . At point D a vertically acting, linear viscous damper with the damping constant c and a point mass m is applied. At the midpoint B of the span AC and at the point D vertical harmonic forces $f_B(t) = F \sin(\omega t)$ and $f_D(t) = \frac{3}{8}F \cos(\omega t)$ are acting with signs as defined in the figure. The beam is infinitely stiff against axial vibrations, and only small vertical vibrations from the static equilibrium state are considered.

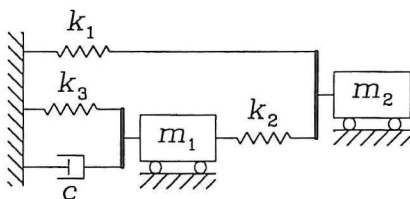
Question 1 (15%, $\mu = 13.3\%$)

Formulate the equation for the motion of the point mass from the static equilibrium state.

Question 2 (10%, $\mu = 3.7\%$)

Determine the stationary motion of the mass, when the excitation has been acting for such period of time that the response from the initial conditions has been dissipated.

PROBLEM 2



The figure shows a plane system of masses, springs and a damper. The masses m_1 and m_2 can only move horizontally in the plane of the system. All springs are linear elastic with the spring constants k_1 , k_2 and k_3 , and are all acting in the direction of motion of the masses. The damper is linear viscous with the damping constant c , and is also acting uni-directionally to the direction of motion of the masses. The connection between the masses, springs and the damper as well as the support and boundary conditions of the system are shown in the figure.

Question 1 (15%, $\mu = 12.6\%$)

Formulate the equations for the motion of the masses from the static equilibrium state.

Question 2 (10%, $\mu = 9.3\%$)

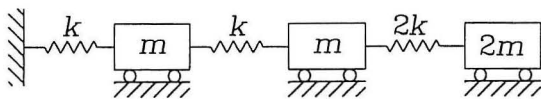
Determine the undamped circular eigenfrequencies and the eigenmodes of the system for the following values of the system parameters

$$m_1 = m_2 = m$$

$$k_1 = k_2 = k_3 = k$$

$$c = 0$$

PROBLEM 3



The figure shows a plane system of masses and springs. The masses with the magnitudes indicated in the figure can only move horizontally in the plane of the system. The springs are all linear elastic with the indicated spring constants. The connection between the masses and the springs as well as the support and boundary conditions of the system is shown in the figure.

Question 1 (15%, $\mu = 14.1\%$)

Formulate the equations for the motion of the masses from the static equilibrium state.

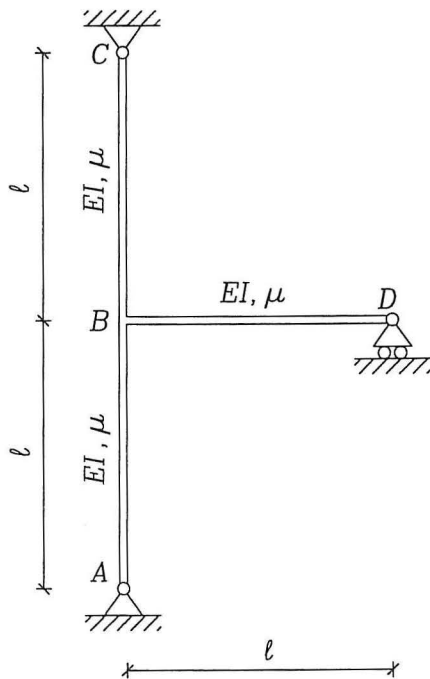
Question 2 (10%, $\mu = 8.2\%$)

Show that the undamped circular eigenfrequencies of the system are given as

$$\omega_j = \begin{cases} 0.37309\sqrt{\frac{k}{m}} & , \quad j = 1 \\ 1.32132\sqrt{\frac{k}{m}} & , \quad j = 2 \\ 2.02852\sqrt{\frac{k}{m}} & , \quad j = 3 \end{cases}$$

Next, determine the undamped eigenmodes of the system.

PROBLEM 4



The figure shows a plane beam structure consisting of the 3 Bernoulli-Euler beams AB , BC and CD all of the length l , with the constant bending stiffness EI and with the constant mass per unit length μ . The beams AB and BC are vertical and the beam BD is horizontal. The structure is rigidly simply supported at the points A and C , movably simply supported at the point D , and the beams AB , BC and BD are rigidly joined at point B . All the sub-beams are assumed to be infinitely stiff against axial deformations. Only small vibrations from the static equilibrium state in the plane of the structure are considered, and the influence on the dynamics from possible axial forces in the static equilibrium state is ignored.

Question 1 (25%, $\mu = 8.6\%$)

Formulate the frequency condition for the determination of the undamped circular eigenfrequencies of the structure.

SOLUTIONS

PROBLEM 1

Question 1:

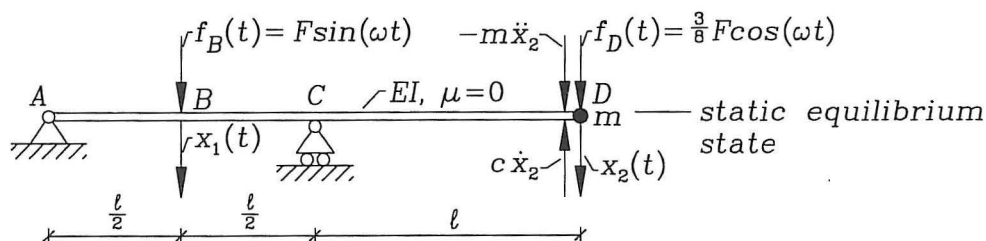


Fig. 1: Forces on the free beam.

The beam $ABCD$ is massless and infinitely stiff against axial elongations. Hence, the system has but a single degree of freedom, which is selected as the vertical displacement $x_2(t)$ of the point mass from the static equilibrium state with sign as defined in fig. 1. Besides, an artificial degree of freedom $x_1(t)$ indicating the vertical displacement of the indirectly acting force $f_B(t)$ is introduced, cf. (2-160). The beam is cut free from the damper, and the damper force $c\dot{x}_2(t)$ is applied as an external load with sign as defined in fig. 1. Besides, the inertial load $-m\ddot{x}_2(t)$ is applied as an external dynamic load acting on the mass according to d'Alembert's principle. The equation of motion for the mass reads

$$x_2(t) = \delta_{12}f_B(t) + \delta_{22}(-m\ddot{x}_2 - c\dot{x}_2 + f_D(t)) \quad (1)$$

The flexibility coefficients are given as, cf. (B-2), (B-3)

$$\delta_{12} = -\frac{1}{16} \frac{l^3}{EI}, \quad \delta_{22} = \frac{2}{3} \frac{l^3}{EI} \quad (2)$$

Inserting (2) into (1) the following equation of motion is obtained

$$m(\ddot{x}_2 + 2\zeta\omega_0\dot{x}_2 + \omega_0^2x_2) = F_1 \sin(\omega t) + F_2 \cos(\omega t) \quad (3)$$

$$\omega_0 = \sqrt{\frac{1}{m\delta_{22}}} = \sqrt{\frac{3EI}{2ml^3}} \quad (4)$$

$$\zeta = \frac{c}{2\omega_0 m} = c\sqrt{\frac{l^3}{6EI m}} \quad (5)$$

$$F_1 = \frac{\delta_{12}}{\delta_{22}} F = -\frac{3}{32} F \quad (6)$$

$$F_2 = \frac{3}{8} F \quad (7)$$

ω_0 and ζ signify the undamped circular eigenfrequency and the damping ratio of the system, respectively. The right-hand side of (3) can be written

$$F_1 \sin(\omega t) + F_2 \cos(\omega t) = F_0 \cos(\omega t - \alpha) = \text{Re} (F_0 e^{-i\alpha} e^{i\omega t}) \quad (8)$$

where

$$\left. \begin{aligned} F_0 \cos \alpha &= F_2 \\ F_0 \sin \alpha &= F_1 \end{aligned} \right\} \Rightarrow$$

$$F_0 = \sqrt{F_1^2 + F_2^2} = \frac{3\sqrt{17}}{32} F \quad (9)$$

$$\tan \alpha = \frac{F_1}{F_2} = -\frac{1}{4} \Rightarrow$$

$$\alpha = -0.24498 \quad (= -15^\circ.596) \quad (10)$$

The equation of motion can then be given on the standard form (2-58) for forced harmonic vibrations of single degree-of-freedom systems

$$\ddot{x}_2 + 2\zeta\omega_0\dot{x}_2 + \omega_0^2 x_2 = \text{Re} \left(\frac{F_0}{m} e^{-i\alpha} e^{i\omega t} \right) \quad (11)$$

Question 2:

The stationary solution to (11) becomes, see (2-64), (2-69), (2-70), (2-71)

$$x_2(t) = X_2 \cos(\omega t - \Psi) \quad (12)$$

$$X_2 = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\zeta^2\omega_0^2\omega^2}} \quad (13)$$

$$\Psi = \Psi_0 + \alpha = \arctan \left(\frac{2\zeta\omega_0\omega}{\omega_0^2 - \omega^2} \right) - \arctan \left(\frac{1}{4} \right) \quad (14)$$

PROBLEM 2

Question 1:

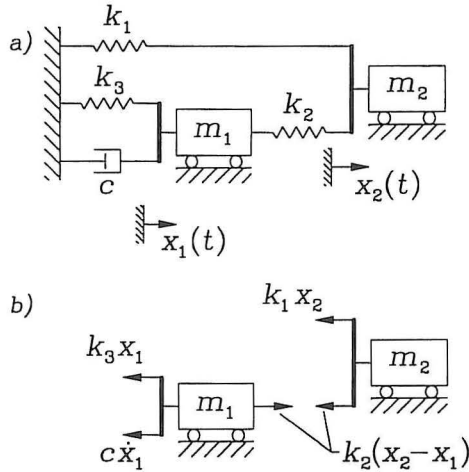


Fig. 1: a) Definition of degrees of freedom. b) Forces on free masses.

The system has 2 degrees of freedom which are selected as the horizontal displacements $x_1(t)$ and $x_2(t)$ of the masses m_1 and m_2 from the static equilibrium state with signs as defined in fig. 1a. The masses are cut free from the springs and the damper, and the spring forces $k_1 x_2(t)$, $k_2(x_2(t) - x_1(t))$, $k_3 x_1(t)$ and the damper force $c \dot{x}_1(t)$ are applied as external forces with signs as shown in fig. 1b. Newton's 2nd law of motion for each of the masses provide

$$\left. \begin{aligned} m_1 \ddot{x}_1 &= k_2(x_2 - x_1) - k_3 x_1 - c \dot{x}_1 \\ m_2 \ddot{x}_2 &= -k_2(x_2 - x_1) - k_1 x_2 \end{aligned} \right\} \Rightarrow$$

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \quad (1)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_2 + k_3 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \quad (2)$$

Question 2:

For $m_1 = m_2 = m$, $k_1 = k_2 = k_3 = k$ the circular eigenfrequencies ω_j and the eigenmodes $\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix}$ become, cf. (3-42)

$$\begin{bmatrix} 2 - \lambda_j & -1 \\ -1 & 2 - \lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3)$$

$$\lambda_j = \frac{m\omega_j^2}{k} \quad (4)$$

The characteristic equation becomes

$$(2 - \lambda_j)(2 - \lambda_j) - 1 \cdot 1 = 0 \Rightarrow$$

$$\lambda_j = \begin{cases} 1 & , \quad j = 1 \\ 3 & , \quad j = 2 \end{cases} \quad (5)$$

$$\omega_j = \begin{cases} 1.0\sqrt{\frac{k}{m}} & , \quad j = 1 \\ \sqrt{3}\sqrt{\frac{k}{m}} & , \quad j = 2 \end{cases} \quad (6)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix} \quad (7)$$

The first component $\Phi_1^{(j)}$ is determined from the 2nd equation of (3)

$$-1 \cdot \Phi_1^{(j)} + (2 - \lambda_j) \cdot 1 = 0 \Rightarrow$$

$$\Phi_1^{(j)} = 2 - \lambda_j = \begin{cases} 1 & , \quad j = 1 \\ -1 & , \quad j = 2 \end{cases} \quad (8)$$

Note:

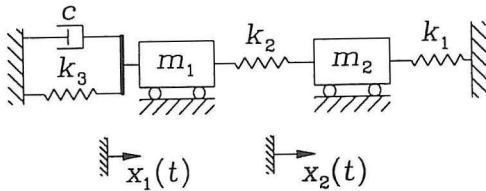


Fig. 2: Equivalent system.

It should be realized that the present system is dynamically equivalent to the one shown in fig. 2. Hence, the equations of motion might have been obtained by specializing the general equations of motion for a 2 degrees-of-freedom system (3-38), (3-39), and eigenfrequencies and eigenmodes might have been obtained by specializing (3-61).

PROBLEM 3

Question 1:

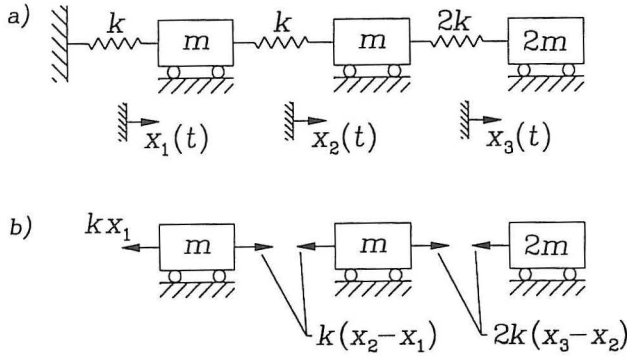


Fig. 1: a) Definition of degrees of freedom. b) Forces on free masses.

The system has 3 degrees of freedom which are selected as the horizontal displacements $x_1(t)$, $x_2(t)$ and $x_3(t)$ of the masses m , m and $2m$ from the static equilibrium state with signs as defined in fig. 1a. The masses are cut free from the springs, and the spring forces $kx_1(t)$, $k(x_2(t) - x_1(t))$ and $2k(x_3(t) - x_2(t))$ are applied as external forces with signs as shown in fig. 1b. Newton's 2nd law of motion for each of the masses provides

$$\left. \begin{aligned} m\ddot{x}_1(t) &= k(x_2 - x_1) - kx_1 \\ m\ddot{x}_2(t) &= 2k(x_3 - x_2) - k(x_2 - x_1) \\ 2m\ddot{x}_3(t) &= -2k(x_3 - x_2) \end{aligned} \right\} \Rightarrow$$

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \quad (1)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2k & -k & 0 \\ -k & 3k & -2k \\ 0 & -2k & 2k \end{bmatrix} \quad (2)$$

Question 2:

The circular eigenfrequencies ω_j and the eigenmodes $\Phi^{(j)T} = [\Phi_1^{(j)}, \Phi_2^{(j)}, \Phi_3^{(j)}]$ then become, cf. (3-42)

$$\begin{bmatrix} 2 - \lambda_j & -1 & 0 \\ -1 & 3 - \lambda_j & -2 \\ 0 & -2 & 2 - 2\lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \\ \Phi_3^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (3)$$

$$\lambda_j = \frac{m\omega_j^2}{k} \quad (4)$$

The characteristic equation becomes

$$(2 - \lambda_j)((3 - \lambda_j)(2 - 2\lambda_j) - 4) + (2 - 2\lambda_j)(-1) = 0 \Rightarrow$$

$$\lambda_j^3 - 6\lambda_j^2 + 8\lambda_j - 1 = 0 \quad (5)$$

From the analytical solution of the 3rd order equation (Cardano's theorem) the solution of (5) is given as

$$\lambda_j = \begin{cases} 2 - \frac{2}{\sqrt{3}} \cos \theta - 2 \sin \theta \simeq 0.1391941 & , \quad j = 1 \\ 2 - \frac{2}{\sqrt{3}} \cos \theta + 2 \sin \theta \simeq 1.7458983 & , \quad j = 2 \\ 2 + \frac{4}{\sqrt{3}} \cos \theta \simeq 4.1149075 & , \quad j = 3 \end{cases} \quad (6)$$

$$\omega_j = \begin{cases} \sqrt{2 - \frac{2}{\sqrt{3}} \cos \theta - 2 \sin \theta} \sqrt{\frac{k}{m}} = 0.3730873 \sqrt{\frac{k}{m}} & , \quad j = 1 \\ \sqrt{2 - \frac{2}{\sqrt{3}} \cos \theta + 2 \sin \theta} \sqrt{\frac{k}{m}} = 1.3213245 \sqrt{\frac{k}{m}} & , \quad j = 2 \\ \sqrt{2 + \frac{4}{\sqrt{3}} \cos \theta} \sqrt{\frac{k}{m}} = 2.0285235 \sqrt{\frac{k}{m}} & , \quad j = 3 \end{cases} \quad (7)$$

where θ is given as

$$\theta = \frac{1}{3} \arctan \left(\sqrt{\frac{229}{27}} \right) \quad (8)$$

In the present solution the circular eigenfrequencies have been found analytically. The assumed approach for proving the statement of the problem is from the given solutions (7) to evaluate the eigenvalues (6) and show that these fulfil the 3rd order equation (5).

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \\ 1 \end{bmatrix} \quad (9)$$

The first and second components $\Phi_1^{(j)}$, $\Phi_2^{(j)}$ are determined from the 2nd and 3rd equations of (3)

$$\left. \begin{aligned} -\Phi_1^{(j)} + (3 - \lambda_j)\Phi_2^{(j)} - 2 \cdot 1 &= 0 \\ -2\Phi_2^{(j)} - (2 - 2\lambda_j) \cdot 1 &= 0 \end{aligned} \right\} \Rightarrow$$

$$\Phi_1^{(j)} = (3 - \lambda_j)(1 - \lambda_j) - 2 = \lambda_j^2 - 4\lambda_j + 1 \quad (10a)$$

$$\Phi_2^{(j)} = 1 - \lambda_j \quad (10b)$$

Inserting the numerical values of (6) the eigenmodes become

$$\Phi^{(1)} = \begin{bmatrix} 0.4625984 \\ 0.8608059 \\ 1 \end{bmatrix}, \quad \Phi^{(2)} = \begin{bmatrix} -2.9354323 \\ -0.7458983 \\ 1 \end{bmatrix}, \quad \Phi^{(3)} = \begin{bmatrix} 1.4728339 \\ -3.1149075 \\ 1 \end{bmatrix} \quad (11)$$

PROBLEM 4

Question 1:

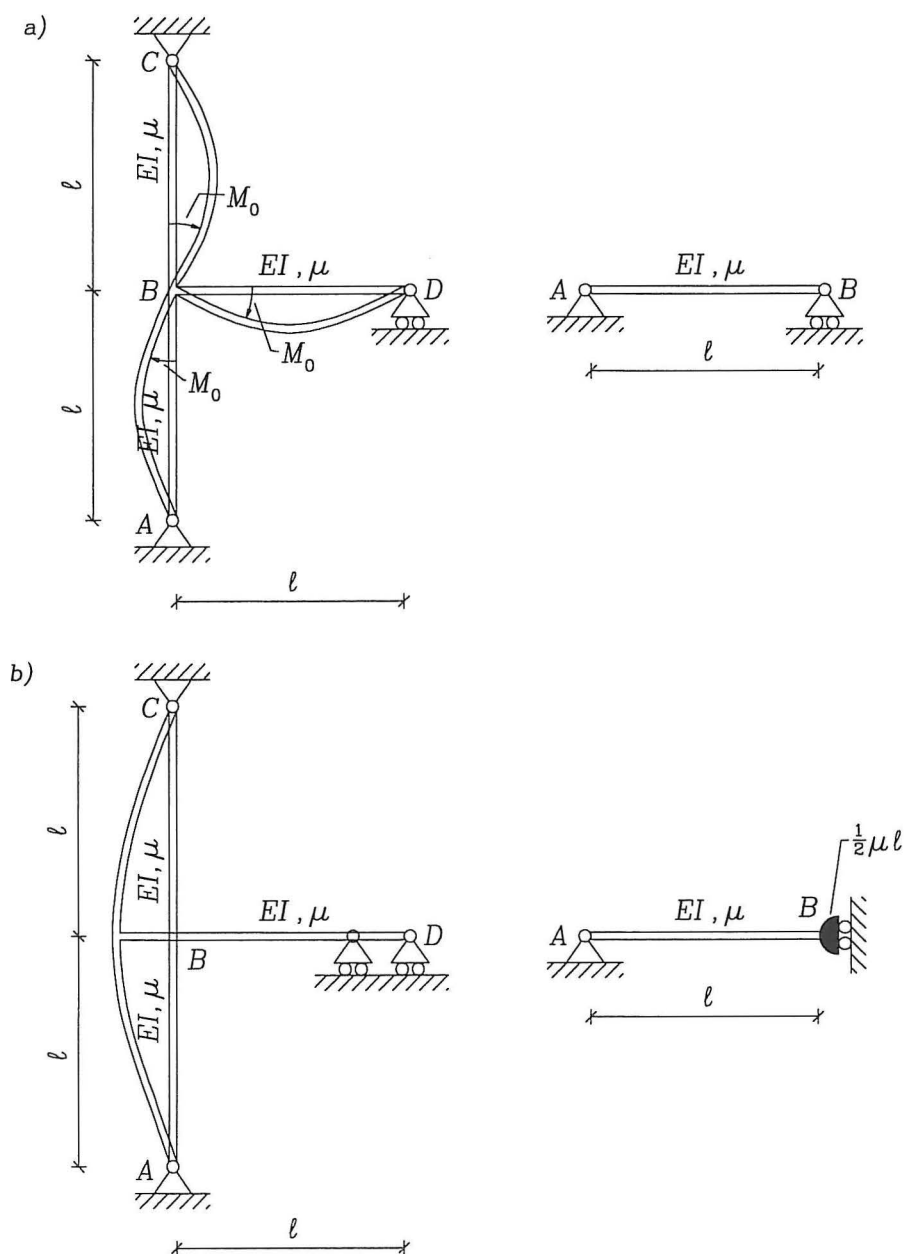


Fig. 1: a) Anti-symmetric eigenvibrations and equivalent system. b) Symmetric eigenvibrations and equivalent system.

The structure is geometrically and mechanically symmetric around the line BD . Hence, the eigenvibrations will separate into anti-symmetric and symmetric eigenvibrations as sketched in figs. 2a and 2b, respectively.

For anti-symmetric eigenvibrations the eigenmodes for each of the 3 sub-beams AB , BC and CD will be congruent. Hence, the amplitudes of the bending moments at the end sections adjacent to mode B will all be equal to the value M_0 , see fig. 1a. If node B is cut free moment equilibrium then requires $M_0 = 0$. During anti-symmetric eigenvibrations all sub-beams then have zero displacement and zero bending moment at node B , which is tantamount to a fixed simple support. Anti-symmetric eigenvibrations can then be analysed by the equivalent simply supported beam shown in fig. 1a. From (4-29) it then follows that

$$\lambda_j = j\pi \quad , \quad j = 2, 4, 6, \dots \quad (1)$$

For symmetric eigenvibrations the node rotation in node B is zero. Beam BC is undeformed and is equivalent to a point mass at point B of magnitude $m = \mu l$. The symmetric eigenvibrations can then be analysed by the equivalent system shown in fig. 1b. The frequency condition for this system follows from lecture 1.9, problem 2, eq. (8) with $\gamma = \frac{\mu l}{2\mu l} = \frac{1}{2}$, i.e.

$$2 + \frac{1}{2}\lambda(\tanh \lambda - \tan \lambda) = 0 \quad (2)$$

The solution to (11) has been given in lecture 1.9, problem 2, eq. (9), and is identical to those in eq. (9) of the present problem for $j = 1, 3, 5, \dots$. It should be emphasized that the general case, where the parameters of the sub-beams AB and BC are not identical, cannot be analysed by the indicated symmetry considerations.

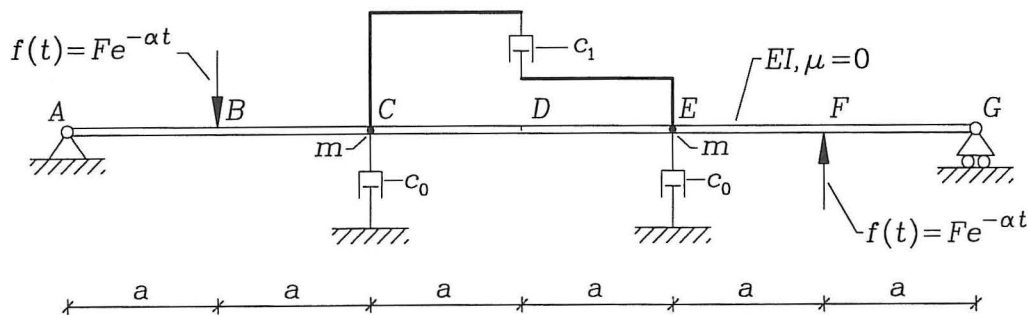
The 6 lowest solutions of (1) and (2) become

$$\lambda_j = \begin{cases} 1.31966 & , \quad j = 1 & \text{(symmetric)} \\ 3.14159 & , \quad j = 2 & \text{(anti-symmetric)} \\ 4.23720 & , \quad j = 3 & \text{(symmetric)} \\ 6.28319 & , \quad j = 4 & \text{(anti-symmetric)} \\ 7.28084 & , \quad j = 5 & \text{(symmetric)} \\ 9.42478 & , \quad j = 6 & \text{(anti-symmetric)} \end{cases} \quad (3)$$

2.14 June 22, 1994

Duration: 4h

PROBLEM 1



The figure shows a horizontal, plane, massless Bernoulli-Euler beam $ABCDEFG$ of the length $6a$ and with the constant bending stiffness EI . The position of the points on the beam is shown in the figure. The beam is simply supported at the points A and G . At the points C and E point masses of the magnitudes m and linear viscous damping elements with the damping constants c are attached. The damping elements are acting in the vertical direction. Between the points C and E an additional linear viscous damping element is attached with the damping constant c_1 , which is activated by the relative motion of the points C and E . The damping element is connected to the points C and E by the indicated gallows-like structures, which are assumed to be infinitely stiff.

At the points B and F the vertical force $f(t) = Fe^{-\alpha t}$, $t \geq 0$, where F and α are positive constants. The force is acting downwards at point B and upwards at point F . The beam is assumed to be infinitely stiff against axial vibrations, and only small vertical motions from the static equilibrium state are considered. The beam is assumed to be at rest at the time $t = 0$.

Question 1 (30%. $\mu = 18.2\%$)

Determine the motion of the masses from the static equilibrium state.

Question 2 (10%. $\mu = 1.9\%$)

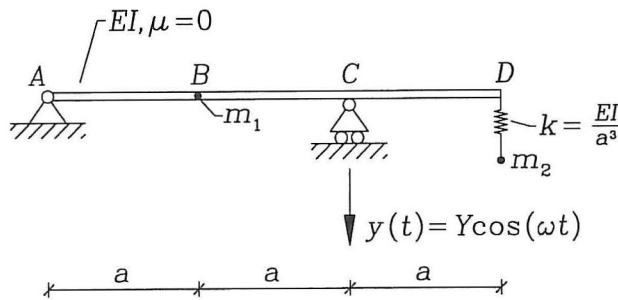
Next, the the vertical forces $f(t) = Fe^{-\alpha t}$ at the points B and F are both assumed to be downward directed. Answer the same question 1 again.

The following definite integral may be of help at the solution of the problem

$$\int_0^t e^{-\zeta\omega_0(t-\tau)} \sin(\omega_d(t-\tau)) e^{-\alpha\tau} d\tau =$$

$$\frac{1}{(\alpha - \zeta\omega_0)^2 + \omega_d^2} \left(e^{-\zeta\omega_0 t} ((\alpha - \zeta\omega_0) \sin(\omega_d t) - \omega_d \cos(\omega_d t)) + \omega_d e^{-\alpha t} \right)$$

PROBLEM 2



The figure shows a horizontal, plane, massless Bernoulli-Euler beam $ABCD$ of the length $3a$, free of damping and with the constant bending stiffness EI . The sub-beams AB , BC and CD all have the same length a . The beam is simply supported at the point A and is extended continuously over the support at point C . At point B a point mass m_1 is attached, and at point D a vertical linear elastic spring with the spring constant $k = \frac{EI}{a^3}$, massless and free of damping, is attached. At the free end of the spring a point mass m_2 is attached. The support at the point C is assumed to move harmonically in the vertical direction as $y(t) = Y \cos(\omega t)$ where Y and ω are real, positive constants, whereas the support at point A is at rest. The beam is assumed to be infinitely stiff against axial deformations, and only small vertical vibrations of the beam and the mass m_2 from the static equilibrium state are considered.

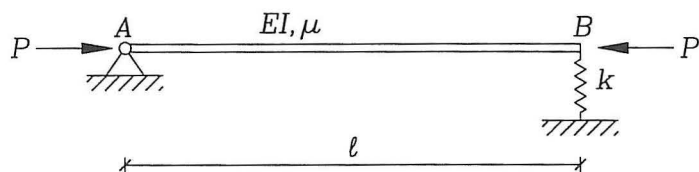
Question 1 (30%. $\mu = 15.9\%$)

Formulate the equations of motion for the determination of the vertical motion of the masses.

Question 2 (10%. $\mu = 3.3\%$)

With the data $m_1 = 2m$ and $m_2 = m$, determine the value of the circular frequency ω of the harmonic motion of the support at point C for which the masses m_1 and m_2 , respectively, are at rest in the stationary motion, as the response from possible initial conditions has died away.

PROBLEM 3



The figure shows a plane, horizontal, rectilinear Bernoulli-Euler beam AB of the length l , free of damping, with the constant bending stiffness EI and with the constant mass per unit length μ . The beam is simply supported at point A and supported at point B by a vertical, linear elastic spring with the spring constant k . In the static equilibrium state the beam is loaded by the compressive axial force P as shown in the figure. The beam is assumed to be infinitely stiff against axial deformations, and only small vertical vibrations from the static equilibrium state are considered.

Question (20%. $\mu = 15.0\%$)

Formulate the frequency condition for the determination of the undamped circular eigenfrequencies of the beam.

SOLUTIONS

PROBLEM 1

Question 1:

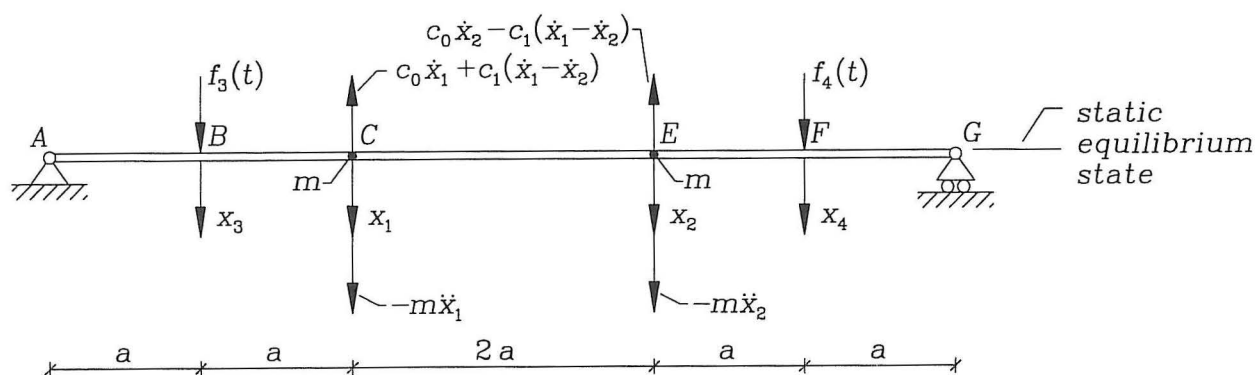


Fig. 1: Forces on free beam.

The beam is massless. Hence, the system has but 2 degrees of freedom, which are selected as the vertical displacements $x_1(t)$ and $x_2(t)$ of the points C and E with signs as defined in fig. 1. Besides, the vertical displacement of the points B and F is introduced as auxiliary degrees of freedom $x_3(t)$ and $x_4(t)$ with the indicated signs. The external dynamic loads $f_3(t)$ and $f_4(t)$ are considered positive when acting in the direction of $x_3(t)$ and $x_4(t)$. The beam is cut free from the dampers, and the damper forces $c_0\dot{x}_1(t) + c_1(\dot{x}_1(t) - \dot{x}_2(t))$ and $c_0\dot{x}_2(t) - c_1(\dot{x}_1(t) - \dot{x}_2(t))$ are applied as external forces at points C and E with signs as defined in fig. 1. Further, the inertial loads $-m\ddot{x}_1(t)$ and $-m\ddot{x}_2(t)$ are applied as external loads in accordance with d'Alembert's principle. The equations of motion then read, cf. (3-1)

$$\left. \begin{aligned} x_1(t) &= \delta_{11}(-m\ddot{x}_1 - c_0\dot{x}_1 - c_1(\dot{x}_1 - \dot{x}_2)) + \delta_{12}(-m\ddot{x}_2 - c_0\dot{x}_2 + c_1(\dot{x}_1 - \dot{x}_2)) + \delta_{13}f_3(t) + \delta_{14}f_4(t) \\ x_2(t) &= \delta_{21}(-m\ddot{x}_1 - c_0\dot{x}_1 - c_1(\dot{x}_1 - \dot{x}_2)) + \delta_{22}(-m\ddot{x}_2 - c_0\dot{x}_2 + c_1(\dot{x}_1 - \dot{x}_2)) + \delta_{23}f_3(t) + \delta_{24}f_4(t) \end{aligned} \right\} \quad (1)$$

The flexibility coefficients are given as, see (B-1)

$$\mathbf{D} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \frac{4}{9} \frac{a^3}{EI} \begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix} \quad (2)$$

$$\mathbf{D}_1 = \begin{bmatrix} \delta_{13} & \delta_{14} \\ \delta_{23} & \delta_{24} \end{bmatrix} = \frac{1}{18} \frac{a^3}{EI} \begin{bmatrix} 38 & 31 \\ 31 & 38 \end{bmatrix} \quad (3)$$

The equations of motion for the displacements of points B and E can then be written

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \quad (4)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_0 + c_1 & -c_1 \\ -c_1 & c_0 + c_1 \end{bmatrix} \quad (5)$$

$$\mathbf{K} = \frac{3}{20} \frac{EI}{a^3} \begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix}, \quad \mathbf{f}(t) = \mathbf{KD}_1 \begin{bmatrix} f_3(t) \\ f_4(t) \end{bmatrix} = \frac{1}{120} \begin{bmatrix} 87 & -18 \\ -18 & 87 \end{bmatrix} \begin{bmatrix} f_3(t) \\ f_4(t) \end{bmatrix} \quad (6)$$

In question (1) the dynamic loading is given as follows

$$\left. \begin{aligned} f_3(t) &= f(t) = Fe^{-\alpha t} \\ f_4(t) &= -f(t) = -Fe^{-\alpha t} \end{aligned} \right\}, \quad t > 0 \quad (7)$$

Because of the anti-symmetric loading and the geometrical and mechanical symmetry of the structure around point D , the displacements become anti-symmetric around point D . Hence

$$x_2(t) = -x_1(t) \quad (8)$$

The 1st equation of (4) then provides

$$\left. \begin{aligned} m(\ddot{x}_1 + 2\zeta\omega_0\dot{x}_1 + \omega_0^2 x_1) &= \frac{7}{8}Fe^{-\alpha t} \quad , \quad t > 0 \\ x_1(0) &= 0 \quad , \quad \dot{x}_1(0) = 0 \end{aligned} \right\} \quad (9)$$

where

$$\omega_0^2 = \frac{9}{4} \frac{EI}{ma^3} \quad (10)$$

$$2\zeta\omega_0 = \frac{c_0 + 2c_1}{m} \quad (11)$$

The displacement of point C then becomes, cf. (2-121)

$$x_1(t) = \int_0^t h(t-\tau) \frac{7}{8} Fe^{-\alpha\tau} d\tau \quad (12)$$

where, cf. (2-49), (2-110)

$$h(u) = \begin{cases} 0 & , \quad u < 0 \\ \frac{1}{m\omega_d} e^{-\zeta\omega_0 u} \sin(\omega_d u) & , \quad u \geq 0 \end{cases} \quad (13)$$

$$\omega_d = \omega_0 \sqrt{1 - \zeta^2} \quad (14)$$

Upon inserting the impulse response functions as given by (13) and evaluating the integral in (12) the following result is obtained

$$\begin{aligned} x_1(t) &= \frac{\frac{7}{8}F}{m\omega_d((\alpha - \zeta\omega_0)^2 + \omega_d^2)} \cdot \\ &\left(e^{-\zeta\omega_0 t} ((\alpha - \zeta\omega_0) \sin(\omega_d t) - \omega_d \cos(\omega_d t)) + \omega_d e^{-\alpha t} \right) \end{aligned} \quad (15)$$

Question 2:

The load is now symmetric around point D , i.e.

$$\left. \begin{aligned} f_3(t) &= f(t) = Fe^{-\alpha t} \\ f_4(t) &= f(t) = Fe^{-\alpha t} \end{aligned} \right\} \quad , \quad t > 0 \quad (16)$$

Hence, the displacements also become symmetric

$$x_2(t) \equiv x_1(t) \quad (17)$$

The 1st equation of (4) then provides

$$\left. \begin{aligned} m(\ddot{x}_1 + 2\zeta\omega_0\dot{x}_1 + \omega_0^2 x_1) &= \frac{23}{40} F e^{-\alpha t} \quad , \quad t > 0 \\ x_1(0) &= 0 \quad , \quad \dot{x}_1(0) = 0 \end{aligned} \right\} \quad (18)$$

where

$$\omega_0^2 = \frac{3}{20} \frac{EI}{ma^3} \quad (19)$$

$$2\zeta\omega_0 = \frac{c_0}{m} \quad (20)$$

Finally, the displacement of point C becomes

$$\begin{aligned} x_1(t) &= \frac{\frac{23}{40} f}{m\omega_d((\alpha - \zeta\omega_0)^2 + \omega_d^2)} \cdot \\ &\quad \left(e^{-\zeta\omega_0 t} ((\alpha - \zeta\omega_0) \sin(\omega_d t) - \omega_d \cos(\omega_d t)) + \omega_d e^{-\alpha t} \right) \end{aligned} \quad (21)$$

Notice that ζ , ω_0 and ω_d have different meanings in (15) and (21).

PROBLEM 2

Question 1:

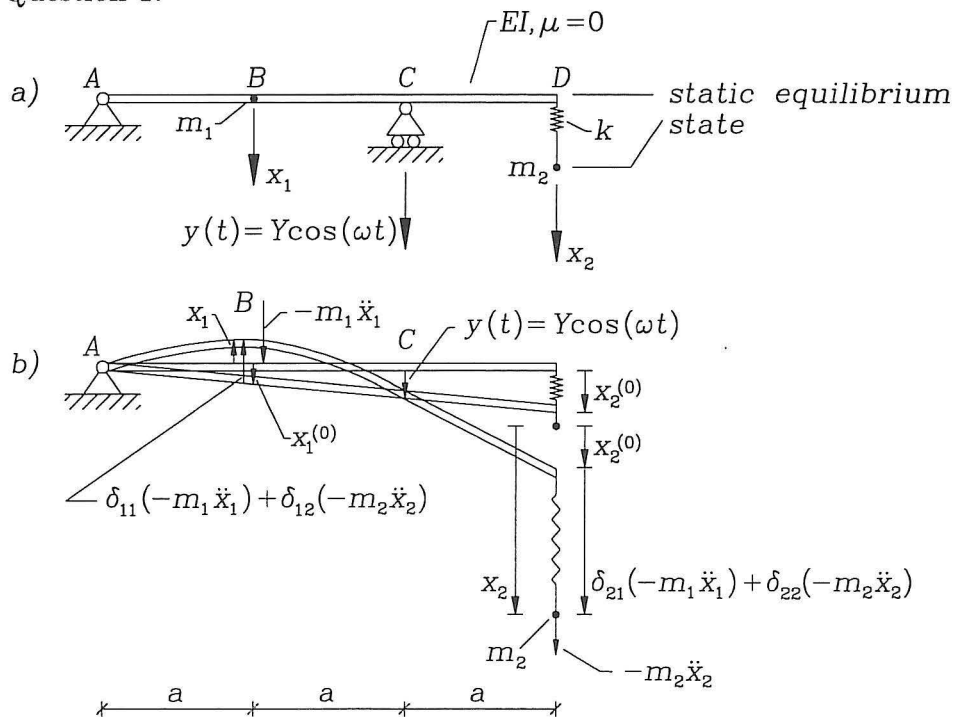


Fig. 1: a) Definition of degrees of freedom. b) Forces on oscillating beam and mass m_2 .

The beam and the spring are massless. Hence, the system has but 2 degrees of freedom which are selected as the vertical displacements $x_1(t)$ and $x_2(t)$ of the point masses m_1 and m_2 from their static equilibrium position with signs as defined in fig. 1a. The displacements of the masses are made up of a quasi-static component $x_1^{(0)}(t)$ from the stiff-body motion of the beam and the spring, and an elastic contribution $\delta_{i1}(-m_1\ddot{x}_1) + \delta_{i2}(-m_2\ddot{x}_2)$ from the inertial loads $-m_1\ddot{x}_1$ and $-m_2\ddot{x}_2$ which are applied with the signs shown in fig. 1b.

The quasi-static motion can be written, cf. (3-327)

$$\mathbf{x}^{(0)}(t) = \begin{bmatrix} x_1^{(0)}(t) \\ x_2^{(0)}(t) \end{bmatrix} = \mathbf{U}y(t) \quad , \quad \mathbf{U} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} \quad (1)$$

δ_{11} , $\delta_{12} = \delta_{21}$, δ_{22} are the flexibility coefficients when the beam is simply supported at the points A and C , as seen from the displacement curve shown in fig. 1b. δ_{22} is made up by a contribution $\frac{1}{k}$ from the extension of the spring, and a contribution $\frac{a^3}{EI}$ from the vertical elastic displacement of point D . The flexibility coefficients are given as, see (B-1), (B-2), (B-3)

$$\mathbf{D} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \frac{a^3}{12EI} \begin{bmatrix} 2 & -3 \\ -3 & 12(1 + \frac{EI}{ka^3}) \end{bmatrix} \quad (2)$$

The equations of motion for the masses m_1 and m_2 then become, cf. (3-328), (3-329)

$$\left. \begin{aligned} x_1(t) &= x_1^{(0)}(t) + \delta_{11}(-m_1\ddot{x}_1) + \delta_{12}(-m_2\ddot{x}_2) \\ x_2(t) &= x_2^{(0)}(t) + \delta_{21}(-m_1\ddot{x}_1) + \delta_{22}(-m_2\ddot{x}_2) \end{aligned} \right\} \Rightarrow$$

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \cos(\omega t) \quad (3)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad , \quad \mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad (4)$$

$$\mathbf{K} = \frac{4}{13} \frac{EI}{a^3} \begin{bmatrix} 24 & 3 \\ 3 & 2 \end{bmatrix} \quad , \quad \mathbf{F} = \mathbf{K}\mathbf{U}Y = \frac{6}{13} \frac{EI Y}{a^3} \begin{bmatrix} 11 \\ 3 \end{bmatrix} \quad (5)$$

where $k = \frac{EI}{a^3}$ has been applied.

Question 2:

The stationary response of (3) becomes, see (3-100), (3-101)

$$\mathbf{x}(t) = \mathbf{X} \cos(\omega t) \quad (6)$$

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = (\mathbf{K} - \omega^2 \mathbf{M})^{-1} \mathbf{F} = \frac{3}{2} \frac{Y}{D} \begin{bmatrix} 2 - \lambda & -3 \\ -3 & 24 - 2\lambda \end{bmatrix} \begin{bmatrix} 11 \\ 3 \end{bmatrix} = \frac{3}{2} \frac{Y}{D} \begin{bmatrix} 13 - 11\lambda \\ 39 - 6\lambda \end{bmatrix} \quad (7)$$

where

$$\lambda = \frac{13}{4} \frac{ma^3}{EI} \omega^2 \quad (8)$$

$$D = (24 - 2\lambda)(2 - \lambda) - 9 \quad (9)$$

The stationary response $\mathbf{x}(t) = \mathbf{X} \cos(\omega t)$ is in phase with the excitation $y(t) = Y \cos(\omega t)$, and the amplitude \mathbf{X} is real, because the structural system is free of damping.

From (7) it follows that $x_1(t) \equiv 0$, if

$$\lambda = \lambda'_1 = \frac{13}{11} \Rightarrow \omega = \omega'_1 = \sqrt{\frac{4}{11}} \sqrt{\frac{EI}{ma^3}} \quad (10)$$

$x_2(t) \equiv 0$ is obtained, if

$$\lambda = \lambda'_2 = \frac{39}{6} \Rightarrow \omega = \omega'_2 = \sqrt{2} \sqrt{\frac{EI}{ma^3}} \quad (11)$$

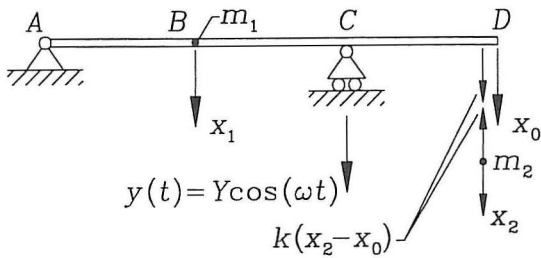


Fig. 2: Forces on oscillating beam and mass m_2 .

Alternatively, the following approach may be used at the solution of the problem. The vertical displacement of point D from the static equilibrium state is introduced as an artificial degree of freedom, see fig. 2. The spring is cut free from the beam, and the

spring force $k(x_2 - x_0)$ is applied as an external force on the beam and on the free mass m_2 with signs as defined in fig. 2. The motion of the points B and D consists of quasi-static stiff-body contributions $x_1^{(0)}(t) = \frac{1}{2}y(t)$ and $x_0^{(0)}(t) = x_2^{(0)}(t) = \frac{3}{2}y(t)$ from the motion of the support at point C and elastic contributions from the inertial force $m_1\ddot{x}_1$ at point B and from the spring force $k(x_2 - x_0)$ at the point D . The equations of motion can then be written, cf. eq. (3)

$$\left. \begin{aligned} x_1(t) &= x_1^{(0)}(t) + \delta_{11}(-m_1\ddot{x}_1) + \delta_{10}(k(x_2 - x_0)) \\ x_0(t) &= x_0^{(0)}(t) + \delta_{01}(-m_1\ddot{x}_1) + \delta_{00}(k(x_2 - x_0)) \end{aligned} \right\} \quad (12)$$

where $\delta_{01} = \delta_{10} = \delta_{12} = -\frac{1}{4}\frac{a^3}{EI}$, $\delta_{00} = \frac{a^3}{EI}$ ($= \delta_{22}$ for $k \rightarrow \infty$). The motion of the free mass m_2 is given as

$$m_2\ddot{x}_2 = -k(x_2 - x_0) \quad (13)$$

The 2nd equation of (12) is solved for $x_0(t)$

$$\begin{aligned} x_0(t) &= \frac{1}{1 + \delta_{00}k} \left(x_0^{(0)}(t) + \delta_{01}(-m_1\ddot{x}_1) + \delta_{00}kx_2 \right) = \\ &= \frac{1}{1 + \frac{ka^3}{EI}} \left(\frac{3}{2}y(t) + \frac{1}{4}\frac{a^3}{EI}m_1\ddot{x}_1 + \frac{ka^3}{EI}x_2 \right) \end{aligned} \quad (14)$$

Next, the equations of motion for $x_1(t)$ and $x_2(t)$ follow from the 1st equation (12) and from (13) upon eliminating $x_0(t)$ by means of (14).

PROBLEM 3

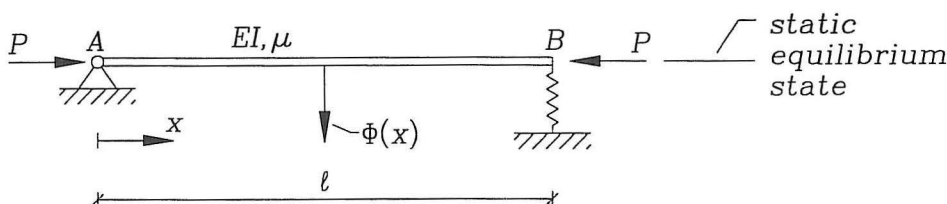


Fig. 1: Beam with constant bending stiffness and constant mass per unit length, loaded with compressive axial force.

The beam AB has constant bending stiffness EI , constant mass per unit length μ , and the compressive axial force $N = -P$. The eigenmodes and associated eigenvalues are

then the solutions to the linear eigenvalue problem, cf. (4.13)

$$\left. \begin{aligned} EI \frac{d^4 \Phi}{dx^4} + P \frac{d^2 \Phi}{dx^2} - \omega^2 \mu \Phi &= 0 \\ \Phi(0) &= 0 \\ \frac{d^2}{dx^2} \Phi(0) &= 0 \\ \frac{d^2}{dx^2} \Phi(l) &= 0 \\ EI \frac{d^3}{dx^3} \Phi(l) + P \frac{d}{dx} \Phi(l) &= k \Phi(l) \end{aligned} \right\} \quad (1)$$

The solution of (1) is given by (4-16), (4-17)

$$\Phi(x) = A \sin\left(\lambda \frac{x}{l}\right) + B \cos\left(\lambda \frac{x}{l}\right) + C \sinh\left(\nu \frac{x}{l}\right) + D \cosh\left(\nu \frac{x}{l}\right) \quad (2)$$

$$\left. \begin{aligned} \lambda^4 - \frac{Pl^2}{EI} \lambda^2 - \frac{\mu\omega^2 l^4}{EI} &= 0 \\ \nu^4 + \frac{Pl^2}{EI} \nu^2 - \frac{\mu\omega^2 l^4}{EI} &= 0 \end{aligned} \right\} \Rightarrow$$

$$\left. \begin{aligned} \lambda &= \sqrt{\sqrt{\frac{\alpha^2}{4} + \lambda_0^4} + \frac{\alpha}{2}} \\ \nu &= \sqrt{\sqrt{\frac{\alpha^2}{4} + \lambda_0^4} - \frac{\alpha}{2}} \end{aligned} \right\} \quad (3)$$

where

$$\alpha = \frac{Pl^2}{EI} \quad (4)$$

$$\lambda_0^4 = \frac{\mu\omega^2 l^4}{EI} \quad (5)$$

The boundary conditions at $x = 0$ imply $B = D = 0$, see (4-24). Then (2) reduces to

$$\Phi(x) = A \sin\left(\lambda \frac{x}{l}\right) + C \sinh\left(\nu \frac{x}{l}\right) \quad (6)$$

Upon inserting (6) into the boundary conditions at $x = l$ the following homogeneous equations are obtained for the determination of A and C

$$\left. \begin{aligned}
& \frac{EI}{l^2} \left(-A\lambda^2 \sin \lambda + C\nu^2 \sinh \nu \right) = 0 \\
& \frac{EI}{l^3} \left(-A\lambda^3 \cos \lambda + C\nu^3 \cosh \nu \right) + \frac{P}{l} \left(A\lambda \cos \lambda + C\nu \cosh \nu \right) = k \left(A \sin \lambda + C \sinh \nu \right)
\end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} -\lambda^2 \sin \lambda & \nu^2 \sinh \nu \\ (-\lambda^3 + \alpha\lambda) \cos \lambda - \kappa \sin \lambda & (\nu^3 + \alpha\nu) \cosh \nu - \kappa \sinh \nu \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (7)$$

where

$$\kappa = \frac{kl^3}{EI} \quad (8)$$

The frequency condition then becomes

$$\begin{aligned}
& -\lambda^2 \sin \lambda ((\nu^3 + \alpha\nu) \cosh \nu - \kappa \sinh \nu) - \\
& \nu^2 \sinh \nu ((-\lambda^3 + \alpha\lambda) \cos \lambda - \kappa \sin \lambda) = 0 \Rightarrow \\
& \nu^2 (\lambda^3 - \alpha\lambda) \cot \lambda - \lambda^2 (\nu^3 + \alpha\nu) \coth \nu + \kappa (\lambda^2 + \nu^2) = 0
\end{aligned} \quad (9)$$

In order to get an overview of the eigenvalues the static buckling loads must first be determined. These are obtained from the eigenvalue problem (1) for $\omega = 0$. The solution of the differential equation becomes

$$\Psi(x) = A \sin \left(\sqrt{\alpha} \frac{x}{l} \right) + B \cos \left(\sqrt{\alpha} \frac{x}{l} \right) + C \frac{x}{l} + D \quad (10)$$

where α is given by (4). The boundary conditions at $x = 0$ imply $B = D = 0$. Then (10) reduces to

$$\Psi(x) = A \sin \left(\sqrt{\alpha} \frac{x}{l} \right) + C \frac{x}{l} \quad (11)$$

The boundary conditions at $x = l$ imply the following homogeneous equations for the determination of A and C

$$\left. \begin{aligned}
& -A EI \frac{\alpha}{l^2} \sin \sqrt{\alpha} = 0 \\
& -A EI \frac{\alpha \sqrt{\alpha}}{l^3} \cos \sqrt{\alpha} + P \left(A \frac{\sqrt{\alpha}}{l} \cos \sqrt{\alpha} + C \frac{1}{l} \right) = k (A \sin \sqrt{\alpha} + C)
\end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} \sin \sqrt{\alpha} & 0 \\ \kappa \sin \sqrt{\alpha} & \kappa - \alpha \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (12)$$

where P and k have been eliminated by means of (4) and (8). Critical values of α , and hence of the compressive axial force, are found if the determinant of the coefficient matrix is equal to 0, i.e.

$$\begin{aligned} \sin \sqrt{\alpha} (\kappa - \alpha) = 0 & \Rightarrow \\ \left. \begin{aligned} \sin \sqrt{\alpha} = 0 \\ \alpha = \kappa \end{aligned} \right\} & \Rightarrow \\ \left. \begin{aligned} \alpha = (j\pi)^2, \quad j = 1, 2, \dots \\ \alpha = \kappa \end{aligned} \right\} & \end{aligned} \quad (13)$$

In case $\alpha = (j\pi)^2 \Leftrightarrow \sin \sqrt{\alpha} = 0$ the second equation of (12) provides $C(\kappa - \alpha) = 0 \Rightarrow C = 0$ (unless $\kappa = (j\pi)^2$!). So, the buckling eigenmodes become

$$\Psi^{(j)}(x) = \sin \left(j\pi \frac{x}{l} \right), \quad j = 1, 2, \dots \quad (14)$$

In case $\alpha = \kappa \Rightarrow \sin \sqrt{\alpha} \neq 0$ (unless $\kappa = (j\pi)^2$) the 1st equation of (12) provides $A \sin \sqrt{\alpha} = 0 \Rightarrow A = 0$. The corresponding buckling eigenmode then becomes

$$\Psi^{(0)}(x) = C \frac{x}{l} \quad (15)$$

(14) corresponds to instability of the beam, whereas the supporting spring is not deformed. (15) corresponds to instability of the supporting spring, leaving the beam undeformed. The eigenmodes (14) and (15) have been illustrated in fig. 2.

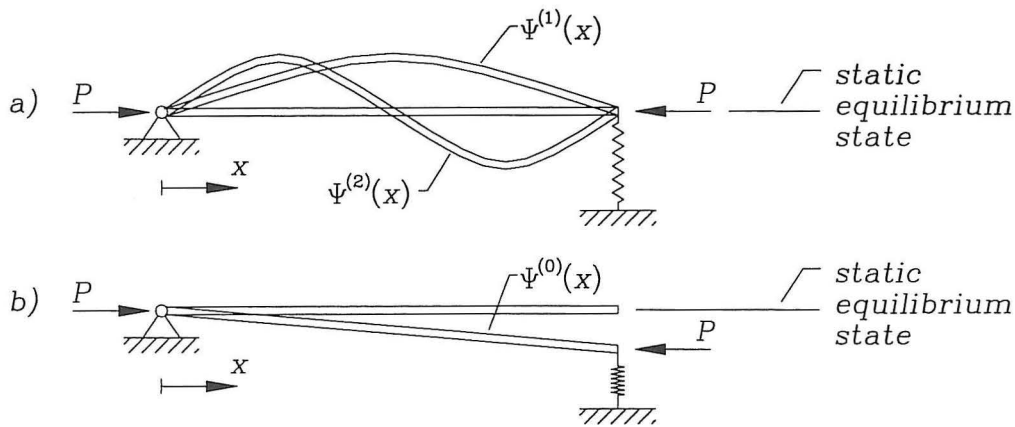


Fig. 2: Statical buckling modes. a) Instability of beam. b) Instability of supporting spring.

Let $\alpha_1 = \min(\pi^2, \kappa)$. As $\alpha \uparrow \alpha_1$ it can be shown from (9) that $\omega_1 \downarrow 0$, where ω_1 is the first circular eigenfrequency. As an example, consider the case $\kappa = 10$, so $\alpha_1 = \pi^2$. Then the following result is obtained from (9) for the 5 lowest eigenvalues of the non-dimensional frequency parameter $\lambda_{0,j}$, $j = 1, \dots, 5$, defined by (5) as a function of the relative magnitude of the compression force $\frac{\alpha}{\alpha_1} = \frac{P}{P_E}$, where $P_E = \pi^2 \frac{EI}{l^2}$ is the Euler instability load.

$\frac{P}{P_E}$	$\lambda_{0,1}$	$\lambda_{0,2}$	$\lambda_{0,3}$	$\lambda_{0,4}$	$\lambda_{0,5}$
0.0	2.23133	4.09539	7.09735	10.21963	13.35598
0.3	2.04351	3.76252	6.94453	10.12466	13.28760
0.6	1.78192	3.28881	6.78058	10.02690	13.21813
0.9	1.28350	2.34871	6.60379	9.92619	13.14754
0.99	0.81946	1.39454	6.54795	9.89537	13.12614
0.999	0.53916	1.05454	6.54229	9.89228	13.12399
0.9999	0.31298	1.00500	6.54172	9.89197	13.12378

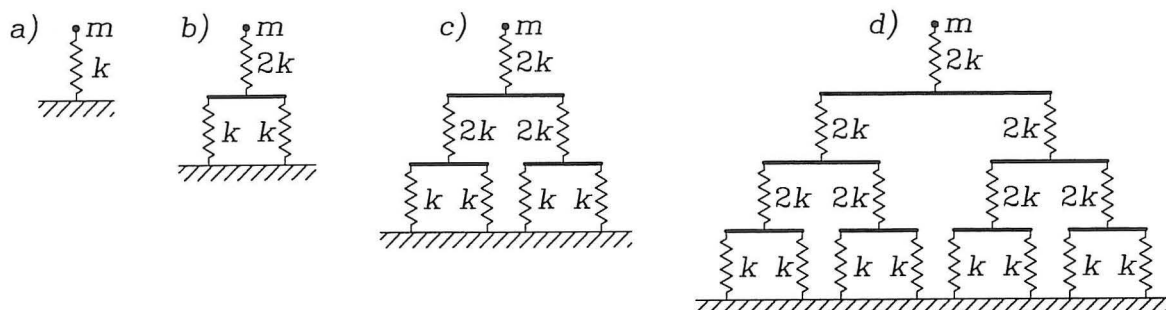
Table 1: 5 lowest eigenvalues of non-dimensional frequency parameter as a function of the compression force.

The values $\lambda_{0,j}$, $j = 1, \dots, 5$, for $\frac{P}{P_E} = 0$ have previously been indicated in lecture 9, problem 1, eq. (10).

2.15 August 24, 1994

Duration: 4h

PROBLEM 1

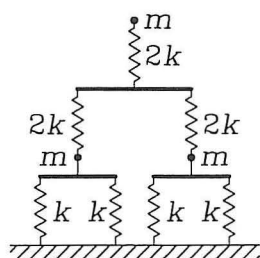


The figure indicates 4 different plane systems made up of springs and masses. The point masses all have the magnitude m and can only move in the vertical direction. All springs are linear elastic with the spring constants indicated in the figure, and all springs are assumed to be free of damping. The springs are coupled via massless, infinitely stiff beams, which have been indicated with a bold line in the figure. The said beams are subjected to such geometrical constraints, that they only can move vertically in the plane of the structure in a way that they remain horizontal during the motion.

Question (20%. $\mu = 19.5\%$)

Determine the circular eigenfrequency for each of the 4 systems.

PROBLEM 2

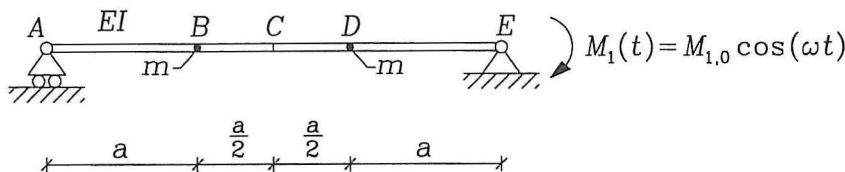


The figure shows a plane system made up of springs and masses. The indicated point masses all have the magnitude m , and can only move in the vertical direction. All springs are linear elastic with the spring constants indicated in the figure, and are assumed massless and free of damping. The springs are coupled via massless, infinitely stiff beams, which have been indicated with a bold line in the figure. The said beams are subjected to such geometrical constraints, that they only can move vertically in the plane of the structure in a way that they remain horizontal during the motion.

Question (25%. $\mu = 13.6\%$)

Determine the circular eigenfrequencies and the eigenmodes of the structure.

PROBLEM 3



The figure shows a horizontal, plane, massless Bernoulli-Euler beam $ABCDE$ of the total length $3a$ and with the constant bending stiffness EI . The beam is simply supported at the points A and E . At the third points B and D , point masses of magnitude m are attached. At point E an external harmonically varying moment $M_1(t) = M_{1,0} \cos(\omega t)$ is acting with the amplitude $M_{1,0}$ and the circular frequency ω . The sign of the moment is defined in the figure. The beam is assumed to be infinitely stiff against axial deformations, and only small vibrations from the static equilibrium state are considered.

Question 1 (20%. $\mu = 13.4\%$)

Determine the equations of motion of the masses from the static equilibrium state, and formulate an equation for the determination of the dynamic bending moment at point C in the middle of the beam.

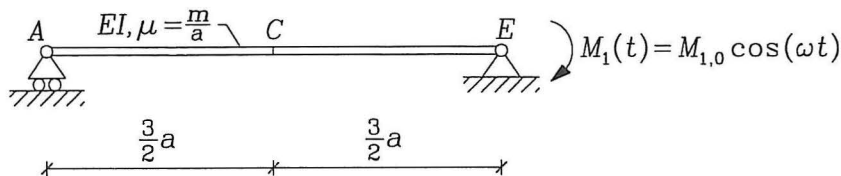
Question 2 (10%. $\mu = 4.4\%$)

Determine the stationary variation with time of the bending moment at point C , when the response from possible initial conditions has died away. Besides, the beam is considered free of damping.

Help:

The displacements at the points B and D from a unit moment $M_1 = 1$ at point E is $-\frac{4}{9} \frac{a^2}{EI}$ and $-\frac{5}{9} \frac{a^2}{EI}$, respectively.

PROBLEM 4



The beam in problem 3 is considered again. However, now the mass of the beam is assumed to be continuously distributed with the constant mass per unit length $\mu = \frac{m}{a}$, and the point masses at the points B and D are no longer present. The stiffness of the beam, the length, the load, as well as the assumptions listed in problem 3 are still valid.

Question (25%. $\mu = 10.9\%$)

Determine the stationary variation with time of the bending moment at point C , when the response from possible initial conditions has died away. Besides the beam is considered free of damping.

Help:

The dynamic moment is replaced by a force couple consisting of harmonically varying upward directed force $F(t) = \frac{M_{1,0}}{\varepsilon} \cos(\omega t)$ at the distance ε from point E , and a similar downwards directed force $\bar{F}(t)$ at point E . In the obtained result based on this replacement the limit passing $\varepsilon \rightarrow 0$ is finally performed. At the calculation of the said limit value the following asymptotic result may be useful, $\sin\left(j\pi\left(1 - \frac{\varepsilon}{l}\right)\right) = (-1)^{j+1} j\pi \frac{\varepsilon}{l} + O\left(\left(\frac{\varepsilon}{l}\right)^2\right)$.

SOLUTIONS

PROBLEM 1

Question 1:

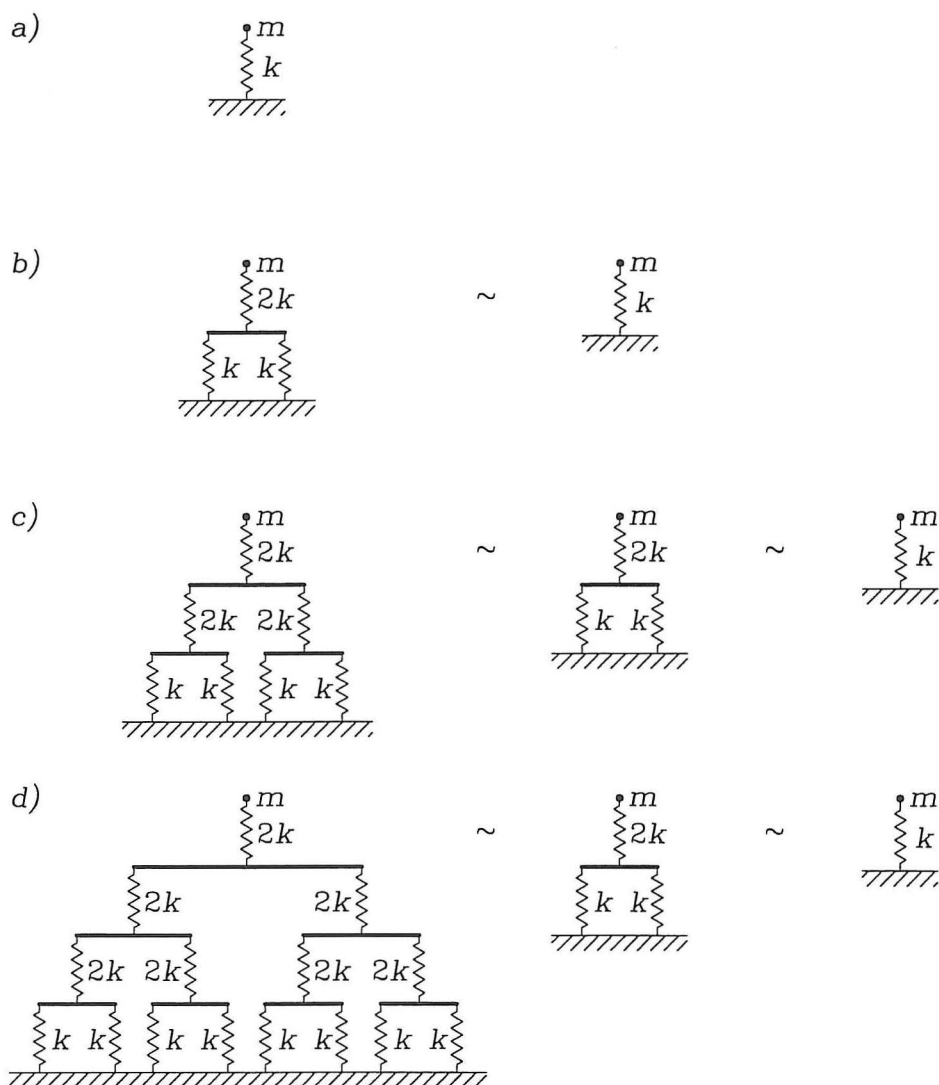


Fig. 1: Equivalent systems.

All 4 systems have but a single degree of freedom which is selected as the vertical displacements of the masses from their static equilibrium state.

Initially, the system b i fig. 1b is analysed. The 2 springs with the spring constant k are parallel. Hence, these can be replaced by a single spring with the spring constant $k_0 = k + k$. The spring system is then equivalent to 2 springs in series, each with the spring constant $2k$. Next, this series system can be replaced by a single spring with the spring constant k_1 determined from $\frac{1}{k_1} = \frac{1}{2k} + \frac{1}{2k} \Rightarrow k_1 = k$. Hence, the system b is elastically equivalent to the system a.

The 2 spring systems in parallel below the upper horizontal beam of the system c in fig. 1c are both identical to the spring system b. Since the latter spring systems have been shown to be equivalent to a single spring with the spring constant $k_1 = k$, system c is elastically equivalent to the 2nd system shown in fig. 1c, which is recognized as system b. Hence, system c is also elastically equivalent to system a.

The 2 spring systems in parallel below the upper horizontal beam of the system d in fig. 1d are both identical to the spring system c. Since the latter spring systems have been shown to be equivalent to a single spring with the spring constant k , system d is elastically equivalent to the 2nd system shown in fig. 1d, which again is recognized as system b. Then system d is also elastically equivalent to system a.

Being elastically equivalent the eigenfrequencies of all 4 systems are then identical. The circular eigenfrequency of system a becomes, cf. (2-7)

$$\omega_0 = \sqrt{\frac{k}{m}} \quad (1)$$

PROBLEM 2

Question 1:

The system has 3 degrees of freedom which are selected as the vertical displacements $x_1(t)$, $x_2(t)$ and $x_3(t)$ of the masses from the static equilibrium state with signs as defined in fig. 1a. The system is mechanically symmetric around the line I-I as shown in fig. 1a. Hence, the eigenvibrations separate into symmetric and anti-symmetric eigenvibrations around the line mentioned.

For symmetric eigenvibrations $x_2(t) \equiv x_3(t)$ and $x_1(t) \neq 0$. The 2 parallel sub-systems below the upper horizontal beam can then be added together to provide the left of the 2 equivalent 2 degrees-of-freedom systems shown in figure 1b. This system can be further reduced by replacing the 2 springs with the spring constants $2k$ and $4k$ in series by a single spring with the spring constant k_1 determined from $\frac{1}{k_1} = \frac{1}{2k} + \frac{1}{4k} \Rightarrow k_1 = \frac{4}{3}k$, cf. (2-27). The final equivalent system has been shown as the second system in fig. 1b.

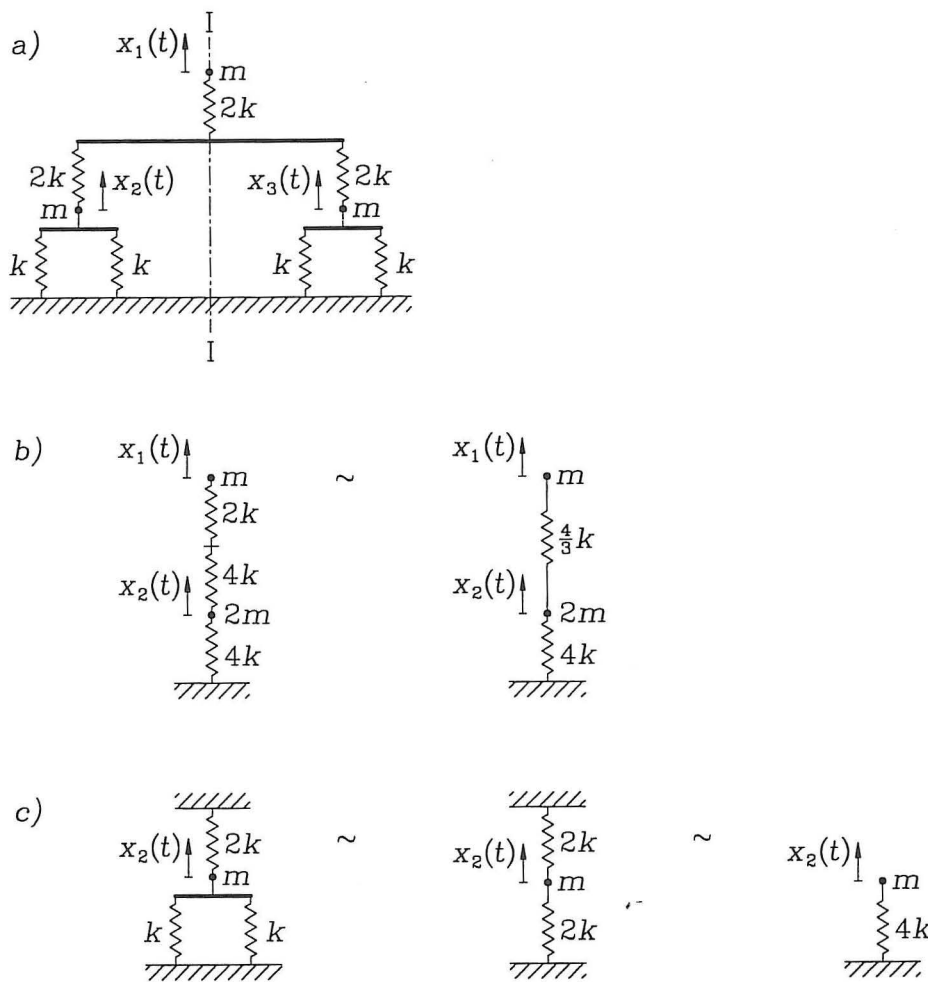


Fig. 1: a) Definition of degrees of freedom. b) Equivalent system for symmetric eigen-vibrations. c) Equivalent system for anti-symmetric eigen-vibrations.

The circular eigenfrequencies of symmetric eigen-vibrations follow from (3-62), using $m_1 = 2m$, $m_2 = m$, $k_1 = 4k$, $k_2 = \frac{4}{3}k$, $k_3 = 0$

$$\omega_j^2 = \begin{cases} \frac{1}{3}(6 - 2\sqrt{3})\frac{k}{m} & , \quad j = 1 \\ \frac{1}{3}(6 + 2\sqrt{3})\frac{k}{m} & , \quad j = 2 \end{cases} \Rightarrow$$

$$\omega_j = \begin{cases} \sqrt{\frac{1}{3}(6 - 2\sqrt{3})}\sqrt{\frac{k}{m}} & , \quad j = 1 \\ \sqrt{\frac{1}{3}(6 + 2\sqrt{3})}\sqrt{\frac{k}{m}} & , \quad j = 2 \end{cases} \quad (1)$$

The undamped eigenmodes become

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \\ 1 \end{bmatrix} \quad , \quad j = 1, 2 \quad (2)$$

where $\Phi_1^{(j)}$ is determined from the 2nd equation of (3-61)

$$\Phi_1^{(j)} = \frac{\frac{4}{3}k}{\frac{4}{3}k - \omega_j^2 m} = \begin{cases} 1 + \sqrt{3} & , \quad j = 1 \\ 1 - \sqrt{3} & , \quad j = 2 \end{cases} \quad (3)$$

For anti-symmetric eigenvibrations $x_2(t) \equiv -x_3(t)$ and $x_1(t) \equiv 0$. The upper horizontal beam is then at rest under anti-symmetric eigenvibrations, and the motion of each of the sub-systems below the beam can be analysed by the equivalent single-degree-of-freedom systems shown in fig. 1c. The circular eigenfrequency of anti-symmetric eigenvibrations becomes, cf. (2-7)

$$\omega_3 = \sqrt{\frac{4k}{m}} = 2\sqrt{\frac{k}{m}} \quad (4)$$

The corresponding undamped eigenmode becomes

$$\Phi^{(3)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad (5)$$

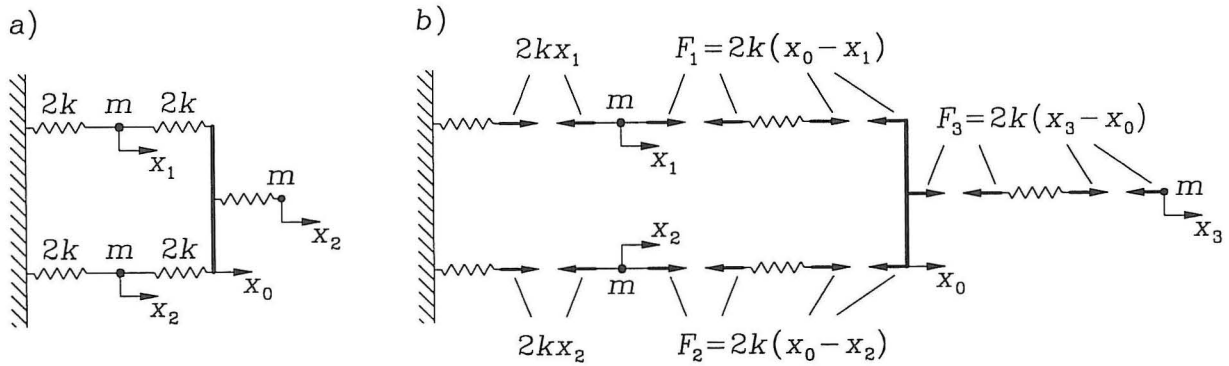


Fig. 2: a) Definition of degrees of freedom. b) Forces on the free masses.

The system has three degrees of freedom, which are selected as the displacement of the masses from the static equilibrium state with signs as shown in fig. 2a. Besides the motion of the infinitely stiff beam is introduced as an artificial degree of freedom x_0 , see fig. 2a. The masses and the infinitely stiff beam are cut free from the springs, and the spring forces are applied as external forces. Newton's 2nd law of motion for each of the three masses then provides

$$\left. \begin{aligned} m\ddot{x}_1 &= F_1 - 2kx_1 \\ m\ddot{x}_2 &= F_2 - 2kx_2 \\ m\ddot{x}_3 &= -F_3 \end{aligned} \right\} \quad (6)$$

where F_1 , F_2 and F_3 signify the spring forces affecting the infinitely stiff beam. These become, see fig. 2b

$$\left. \begin{aligned} F_1 &= 2k(x_0 - x_1) \\ F_2 &= 2k(x_0 - x_2) \\ F_3 &= 2k(x_3 - x_0) \end{aligned} \right\} \quad (7)$$

The infinitely stiff beam must be in equilibrium, i.e.

$$F_1 + F_2 = F_3 \quad (8)$$

Insertion of (7) into (8) provides the following relation for the the auxiliary degree of freedom x_0

$$x_0 = \frac{1}{3}(x_1 + x_2 + x_3) \quad (9)$$

Insertion of (7) and (9) into (6) provides the following equations of motion

$$\left. \begin{aligned} m\ddot{x}_1 &= 2k\left(\frac{1}{3}(x_1 + x_2 + x_3) - x_1\right) - 2kx_1 = \frac{2k}{3}(-5x_1 + x_2 + x_3) \\ m\ddot{x}_2 &= 2k\left(\frac{1}{3}(x_1 + x_2 + x_3) - x_2\right) - 2kx_2 = \frac{2k}{3}(x_1 - 5x_2 + x_3) \\ m\ddot{x}_3 &= -2k\left(x_3 - \frac{1}{3}(x_1 + x_2 + x_3)\right) = \frac{2k}{3}(x_1 + x_2 - 2x_3) \end{aligned} \right\} \quad (10)$$

(10) can be written in the following matrix form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0 \quad (11)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, \quad \mathbf{K} = \frac{2k}{3} \begin{bmatrix} 5 & -1 & -1 \\ -1 & 5 & -2 \\ -1 & -1 & 2 \end{bmatrix} \quad (12)$$

Circular eigenfrequencies and undamped eigenmodes are then determined from the following eigenvalue problem, cf. (3-42)

$$\begin{bmatrix} 5 - \lambda_j & -1 & -1 \\ -1 & 5 - \lambda_j & -2 \\ -1 & -1 & 2 - \lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \\ \Phi_3^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \lambda_j = \frac{3}{2} \frac{m\omega_j^2}{k}, \quad j = 1, 2, 3 \quad (13)$$

Further solution of the eigenvalue problem (13) will not be attempted.

PROBLEM 3

Question 1:

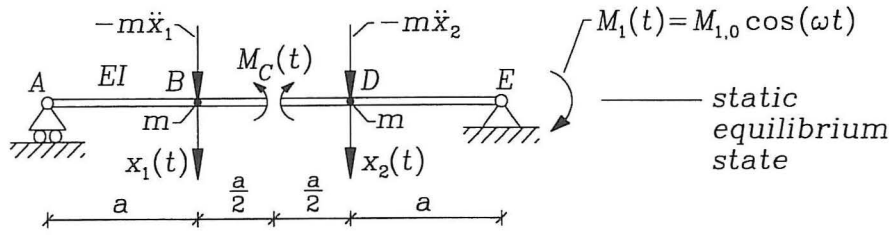


Fig. 1: Definition of signs and loads on the beam.

The beam is massless. Hence, the system has but 2 degrees of freedom which are selected as the vertical displacements $x_1(t)$ and $x_2(t)$ of the points B and D from the static equilibrium state with signs as defined in fig. 1. The dynamic bending moment at the mid-point C is designated $M_C(t)$, and the sign has been defined in fig. 1. Next, the inertial loads $-m\ddot{x}_1(t)$ and $-m\ddot{x}_2(t)$ are applied as external loads at the points B and D according to d'Alembert's principle. The equations of motion for the masses and the equation for $M_C(t)$ then become, cf. (3-1)

$$\left. \begin{aligned} x_1(t) &= \delta_{1M_1} M_1(t) + \delta_{11}(-m\ddot{x}_1) + \delta_{12}(-m\ddot{x}_2) \\ x_2(t) &= \delta_{2M_1} M_1(t) + \delta_{21}(-m\ddot{x}_1) + \delta_{22}(-m\ddot{x}_2) \end{aligned} \right\} \quad (1)$$

$$M_C(t) = \delta_{M_C M_1} M_1(t) + \delta_{M_C 1}(-m\ddot{x}_1) + \delta_{M_C 2}(-m\ddot{x}_2) \quad (2)$$

The flexibility coefficients δ_{ij} are given as, see (B-1)

$$\mathbf{D} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \frac{1}{18} \frac{a^3}{EI} \begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix} \quad (3)$$

The flexibility coefficients δ_{1M_1} and δ_{2M_1} signify the displacements of the points B and D from an external bending moment $M_1 = 1$ at point E. Using standard static analysis techniques these can be found as follows

$$\begin{bmatrix} \delta_{1M_1} \\ \delta_{2M_1} \end{bmatrix} = -\frac{1}{9} \frac{a^2}{EI} \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad (4)$$

The coefficients of influence $\delta_{M_C M_1}$ and $\delta_{M_C j}$ signify the bending moment at point C from an external bending moment $M_1 = 1$ at the point E and a unit load acting uni-directional to the j th degree of freedom. These are easily found to be

$$\begin{bmatrix} \delta_{M_C M_1} \\ \delta_{M_C 1} \\ \delta_{M_C 2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ a \\ a \end{bmatrix} \quad (5)$$

The equations of motion (1) and equation (2) can then be written

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \cos(\omega t) \quad (6)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad (7)$$

$$\mathbf{K} = \mathbf{D}^{-1} = \frac{6}{5} \frac{EI}{a^3} \begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix}, \quad \mathbf{F} = \mathbf{K} \begin{bmatrix} \delta_{1M_1} \\ \delta_{2M_1} \end{bmatrix} M_{1,0} = \frac{2}{5} \frac{M_{1,0}}{a} \begin{bmatrix} 1 \\ -4 \end{bmatrix} \quad (8)$$

$$M_C(t) = -\frac{M_{1,0}}{2} \cos(\omega t) - \frac{a}{2} m (\ddot{x}_1(t) + \ddot{x}_2(t)) \quad (9)$$

Question 2:

The stationary response of the masses of the system (6) is given as, cf. (3-100), (3-101)

$$\mathbf{x}(t) = \mathbf{X} \cos(\omega t) \quad (10)$$

$$\mathbf{X} = \mathbf{H}(\omega) \mathbf{F} \quad (11)$$

$$\mathbf{H}(\omega) = (\mathbf{K} - \omega^2 \mathbf{M})^{-1} = \frac{5}{6(\alpha^2 - 16\alpha + 15)} \begin{bmatrix} 8 - \alpha & 7 \\ 7 & 8 - \alpha \end{bmatrix} \frac{a^3}{EI} \quad (12)$$

$$\alpha = \frac{5}{6} \frac{\omega^2 m a^3}{EI} \quad (13)$$

$\mathbf{x}(t) = \mathbf{X} \cos(\omega t)$ is in phase with the excitation, $\mathbf{f}(t) = \mathbf{F} \cos(\omega t)$, and \mathbf{X} is real, because the structural system is free of damping. By inserting (8) and (12) into (11) the following solution is obtained for the amplitude vector

$$\mathbf{X} = -\frac{1}{3(\alpha^2 - 16\alpha + 15)} \begin{bmatrix} 20 + \alpha \\ 25 - 4\alpha \end{bmatrix} \frac{M_{1,0} a^2}{EI} \quad (14)$$

Upon inserting (10), (14) into (9) the following solution is obtained for the bending moment at point C

$$M_C(t) = -\frac{1}{2} M_{1,0} D_1(\omega) \cos(\omega t) \quad (15)$$

$$D_1(\omega) = 1 + \frac{(45 - 3\alpha)}{3(\alpha^2 - 16\alpha + 15)} \frac{\omega^2 m a^3}{EI} = 1 + \frac{6}{5} \frac{\alpha}{1 - \alpha} \quad (16)$$

where the frequency parameter α is given by (13). $D_1(\omega)$ is a dynamic amplification factor to the quasi-static bending moment $M_C(t) = -\frac{1}{2}M_{1,0} \cos(\omega t)$. The circular eigenfrequencies of the system are determined from $\alpha^2 - 16\alpha + 15 = 0 \Rightarrow \alpha_1 = 1, \alpha_2 = 15$. As seen, $D_1(\omega) \rightarrow \pm\infty$ for $\alpha \rightarrow \alpha_1$, whereas $D_1(\omega)$ approaches the finite value $-\frac{2}{7}$ in the 2nd resonance region as $\alpha \rightarrow \alpha_2$. This is so, because the 2nd eigenmode is anti-symmetric around point C . Hence, the modal contribution to the bending moment from the 2nd mode is 0.

PROBLEM 4

Question 1:

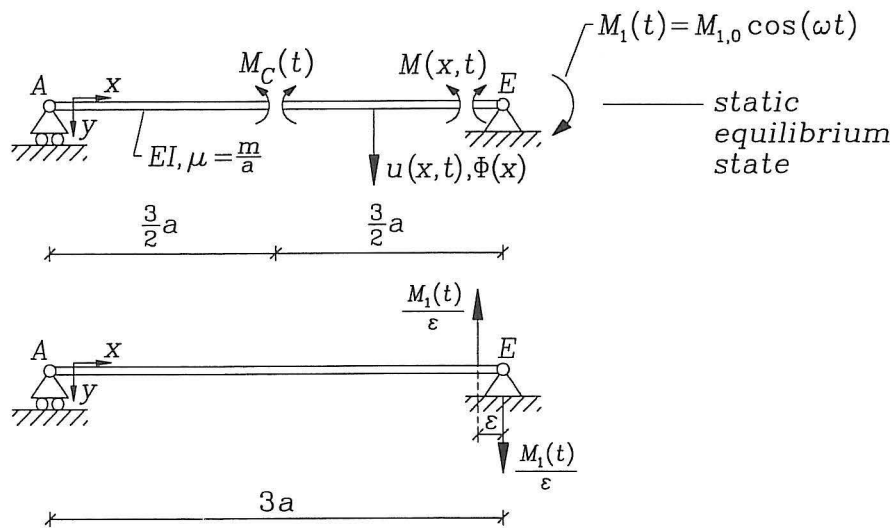


Fig. 1: a) Definition of signs. b) Equivalent representation of the end-section moment $M_1(t)$ in terms of a couple of forces.

A local (x, y) -coordinate system is introduced as shown in fig. 1a. Signs for the vertical displacement from the static equilibrium state $u(x, t)$ and the dynamic bending moment have also been defined in the figure.

The idea is to determine the bending moment $M_C(t)$ at the mid-point by means of modal decomposition technique. Unfortunately, the evaluation of modal loads from end-section moments have not been considered in the course. For this reason the trick shown in fig. 1b becomes necessary, where the end-section moment is represented by a couple of forces of magnitude $\frac{M_1(t)}{\epsilon}$ placed at $x = 3a - \epsilon$ and $x = 3a$ with signs as shown in fig. 1b. The force at $x = 3a$ is transferred directly to the support without any effect on the dynamics of the beam, and can be disregarded in what follows.

The undamped circular eigenfrequencies become, see (4-33)

$$\omega_j = j^2 \pi^2 \sqrt{\frac{EI}{\mu(3a)^4}} = \frac{j^2 \pi^2}{9} \sqrt{\frac{EI}{ma^3}} \quad , \quad j = 1, 2, \dots \quad (1)$$

The eigenmodes become, see (4-31)

$$\Phi(x) = \sin\left(j\pi \frac{x}{3a}\right) \quad , \quad j = 1, 2, \dots \quad (2)$$

With the normalization of the eigenmodes as follows from (2) the modal masses become, cf. (4-64), (4-65)

$$M_j = \int_0^{3a} \mu (\Phi^{(j)}(x))^2 dx = \frac{1}{2} \mu 3a = \frac{3}{2} m \quad , \quad j = 1, 2, \dots \quad (3)$$

The vertical displacement from the static equilibrium state, and the dynamic bending moment $M_C(t)$ become, cf. (4-52), (4-61)

$$u(x, t) = \sum_{j=1}^{\infty} \Phi^{(j)}(x) q_j(t) \quad (4)$$

$$M_C(t) = \sum_{j=1}^{\infty} -EI \frac{d^2}{dx^2} \Phi^{(j)}\left(\frac{3a}{2}\right) q_j(t) = \frac{EI}{9a^2} \sum_{j=1}^{\infty} (j\pi)^2 \sin\left(\frac{j\pi}{2}\right) q_j(t) \quad (5)$$

Since the system is free of damping the modal coordinates are obtained as the solution of the decoupled differential equations, see (4-53)

$$\ddot{q}_j + \omega_j^2 q_j = \frac{1}{M_j} F_j(t) \quad , \quad j = 1, 2, \dots \quad (6)$$

The concentrated load in the negative y -direction at the position $x = 3a - \varepsilon$, representing the end-section moment $M_1(t)$, can formally be written as the following load per unit length, cf. (4-68)

$$f_d(x, t) = -\frac{M_1(t)}{\varepsilon} \delta(x - (3a - \varepsilon)) \quad (7)$$

The modal loads then follow from (4-54) and (2)

$$F_j(t) = \lim_{\varepsilon \rightarrow 0} \int_0^{3a} \Phi^{(j)}(x) f_d(x, t) dx = -\lim_{\varepsilon \rightarrow 0} \int_0^{3a} \sin\left(j\pi \frac{x}{3a}\right) \frac{M_1(t)}{\varepsilon} \delta(x - (3a - \varepsilon)) dx =$$

$$\lim_{\varepsilon \rightarrow 0} \left(-\frac{M_1(t)}{\varepsilon} \sin \left(j\pi \frac{3a - \varepsilon}{3a} \right) \right) = \lim_{\varepsilon \rightarrow 0} \left((-1)^j \frac{j\pi}{3a} M_1(t) + O\left(\frac{\varepsilon}{l}\right) \right) =$$

$$(-1)^j \frac{j\pi}{3a} M_1(t) = (-1)^j \frac{j\pi}{3a} M_{1,0} \cos(\omega t) \quad (8)$$

In the derivation of (8) the 1st order Taylor expansion $\sin(j\pi(1 - \frac{\varepsilon}{l})) = \sin(j\pi) - \cos(j\pi)j\pi\frac{\varepsilon}{l} + O((\frac{\varepsilon}{l})^2) = (-1)^{j+1}j\pi\frac{\varepsilon}{l} + O((\frac{\varepsilon}{l})^2)$ has been used. With $F_j(t)$ given by (8) and use of (3), the stationary solution of (6) becomes

$$q_j(t) = Q_j \cos(\omega t) \quad (9)$$

$$Q_j = (-1)^j \frac{2}{9} (j\pi) \frac{M_{1,0}}{ma} \frac{1}{\omega_j^2 - \omega^2} \quad , \quad j = 1, 2, \dots \quad (10)$$

Upon inserting (9) into (5) the stationary dynamic bending moment at the mid-point is finally obtained

$$M_C(t) = -\frac{1}{2} M_{1,0} D_2(\omega) \cos(\omega t) \quad (11)$$

$$D_2(\omega) = -2 \frac{EI}{9a^2} \sum_{j=1}^{\infty} (j\pi)^2 \sin\left(\frac{j\pi}{2}\right) (-1)^j \frac{2}{9} (j\pi) \frac{1}{ma} \frac{1}{\omega_j^2 - \omega^2} =$$

$$\frac{4}{\pi} \frac{\pi^4}{81} \frac{EI}{ma^3} \sum_{j=1,3,5,\dots}^{\infty} j^3 (-1)^{\frac{j-1}{2}} \frac{1}{j^4 \omega_1^2 - \omega^2} =$$

$$\frac{4}{\pi} \sum_{j=1,3,5,\dots}^{\infty} (-1)^{\frac{j-1}{2}} \frac{j^3}{j^4 - \frac{\omega^2}{\omega_1^2}} = \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(2n-1)^3}{(2n-1)^4 - \frac{\omega^2}{\omega_1^2}} \quad (12)$$

where $\omega_1^2 = \frac{\pi^4}{81} \frac{EI}{ma^3}$ has been used to normalize the circular excitation frequency ω . Consider the well-known series expansion

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \quad (13)$$

From (13) it then follows that $D_2(0) = 1$, as expected. Notice that the convergence of the series (13) is very slow. So is the convergence of (12) as shown below.

Alternatively, the bending moment $M_C(t)$ can be obtained on closed form by direct integration of the partial differential equation for the beam element, without any resort to series solutions. The partial differential equation and associated boundary conditions become, cf. (4-7)

$$\left. \begin{aligned} EI \frac{\partial^4 u}{\partial x^4} + \mu \frac{\partial^2 u}{\partial t^2} &= 0 \quad , \quad x \in]0, 3a[\\ u(0, t) &= \frac{\partial^2 u(0, t)}{\partial x^2} = u(3a, t) = 0 \\ -EI \frac{\partial^2 u(3a, t)}{\partial x^2} &= -M_1(t) = -M_{1,0} \cos(\omega t) \end{aligned} \right\} \quad (14)$$

The mechanical boundary condition at $x = 3a$ expresses that the bending moment $M(3a^-, t)$ within the beam must balance the external moment $M_1(t)$. The stationary displacement field must be harmonic. Since the system is free of damping all mass particles must be in phase and in phase with the end-section bending moment $M_1(t) = M_{1,0} \cos(\omega t)$. Hence, the stationary displacement field is given on the form

$$u(x, t) = U(x) \cos(\omega t) \quad (15)$$

Upon inserting (15) into (14) the real amplitude function $U(x)$ is seen to fulfil the boundary condition problem

$$\left. \begin{aligned} EI \frac{d^4}{dx^4} U(x) - \omega^2 \mu U(x) &= 0 \quad , \quad x \in]0, 3a[\\ U(0) &= \frac{d^2}{dx^2} U(0) = U(3a) = 0 \\ EI \frac{d^2}{dx^2} U(3a) &= M_{1,0} \end{aligned} \right\} \quad (16)$$

The solution of the homogeneous differential equation (16) can be written, cf. (4-88), (4-89)

$$U(x) = A \sin\left(\lambda \frac{x}{3a}\right) + B \cos\left(\lambda \frac{x}{3a}\right) + C \sinh\left(\lambda \frac{x}{3a}\right) + D \cosh\left(\lambda \frac{x}{3a}\right) \quad (17)$$

$$\lambda^4 = \frac{\mu \omega^2 (3a)^4}{EI} = 81 \frac{\omega^2 m a^3}{EI} \quad (18)$$

Inserting (17) into the boundary conditions at $x = a$ provides

$$B + D = 0 \quad (19a)$$

$$\left(\frac{\lambda}{3a}\right)^2 (-B + D) = 0 \quad (19b)$$

(19a) and (19b) imply $B = D = 0$. Then (17) reduces to

$$U(x) = A \sin\left(\lambda \frac{x}{3a}\right) + C \sinh\left(\lambda \frac{x}{3a}\right) \quad (20)$$

Upon inserting (20) into the boundary conditions at $x = 3a$ the following linear equations are obtained for the determination of A and C

$$\left. \begin{aligned} A \sin \lambda + C \sinh \lambda &= 0 \\ EI \frac{\lambda^2}{(3a)^2} (-A \sin \lambda + C \sinh \lambda) &= M_{1,0} \\ A &= -\frac{1}{\lambda^2 \sin \lambda} \frac{9}{2} \frac{M_{1,0} a^2}{EI} \\ C &= \frac{1}{\lambda^2 \sinh \lambda} \frac{9}{2} \frac{M_{1,0} a^2}{EI} \end{aligned} \right\} \Rightarrow \quad (21)$$

(20) can then be written

$$U(x) = \frac{9}{2} \frac{1}{\lambda^2} \left(-\frac{\sin(\lambda \frac{x}{3a})}{\sin \lambda} + \frac{\sinh(\lambda \frac{x}{3a})}{\sinh \lambda} \right) \frac{M_{1,0} a^2}{EI} \quad (22)$$

The stationary dynamic bending moment at the mid-point then becomes

$$M_C(t) = -EI \frac{d^2}{dx^2} U\left(\frac{3a}{2}\right) \cos(\omega t) = -\frac{1}{2} M_{1,0} D_2(\omega) \cos(\omega t) \quad (23)$$

$$D_2(\omega) = \left(\frac{\sin \frac{\lambda}{2}}{\sin \lambda} + \frac{\sinh \frac{\lambda}{2}}{\sinh \lambda} \right) = \frac{1}{2} \left(\frac{1}{\cos \frac{\lambda}{2}} + \frac{1}{\cosh \frac{\lambda}{2}} \right) \quad (24)$$

In the last statement of (24) the trigonometric and hyperbolic identities $\sin(2x) = 2 \sin x \cos x$ and $\sinh(2x) = 2 \sinh x \cosh x$ have been applied. (12) is merely a convergent series expansion of the closed form solution (24) for the dynamic amplification factor $D_2(\omega)$.

λ	$D_1(\omega)$	$D_2(\omega)$
0.0000	1.0000	1.0000
2.0000	1.2365	1.2494
3.1399 ($= \sqrt[4]{\frac{486}{5}}$)	$\pm \infty$	592.88
3.1416 ($= \pi$)	-558.04	$\pm \infty$
4.0000	-0.9345	-1.0686
6.2832 ($= 2\pi$)	-0.2798	-0.4596
9.4248 ($= 3\pi$)	-0.2150	$\pm \infty$

Table 1: Dynamic amplification factors for the stationary bending moment at the mid-point for discretized and continuous models.

In terms of λ as given by (18) the dynamic amplification factor $D_1(\omega)$ of problem 2 can be written

$$D_1(\omega) = 1 + \frac{6}{5} \frac{\frac{5}{6} \frac{\omega^2 m a^3}{EI}}{1 - \frac{5}{6} \frac{\omega^2 m a^3}{EI}} = 1 + \frac{\frac{1}{81} \lambda^4}{1 - \frac{5}{6} \frac{1}{81} \lambda^4} = \frac{486 + \lambda^4}{486 - 5\lambda^4} \quad (25)$$

(24) and (25) have been evaluated in table 1 for selected values of the frequency parameter λ . The discretized model considered in problem 2 is only doing well for excitation frequencies below the 1st circular eigenfrequency ($\lambda = \lambda_1 = \pi$). The discretized model predicts the eigenvalue $\lambda_1 = \sqrt[4]{\frac{486}{5}} \simeq 3.1399$ compared to the exact value $\lambda_1 = \pi$.

Number of term N	$\lambda = 2$	$\lambda = 4$
2	1.0982	-1.2207
4	1.1710	-1.1471
8	1.2098	-1.1082
16	1.2296	-1.0885
32	1.2395	-1.0785
64	1.2445	-1.0736
128	1.2469	-1.0711
256	1.2482	-1.0698
512	1.2488	-1.0692
1024	1.2491	-1.0689
2048	1.2493	-1.0688
4096	1.2494	-1.0687

Table 2: Convergence of series expansion (12) of the dynamic amplification factor.

The series expansion (12) is truncated at the N th term. Introducing the frequency parameter λ , this can be written

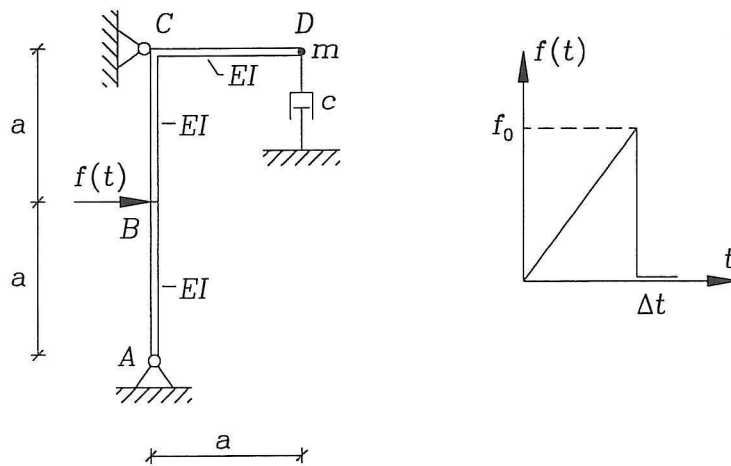
$$D_2(\omega) = \frac{4}{\pi} \sum_{n=1}^N (-1)^{n-1} \frac{(2n-1)^3}{(2n-1)^4 - \frac{\lambda^4}{\pi^4}} \quad (26)$$

For $\lambda = 2$ and $\lambda = 4$ the convergence of the series solution (26) as a function of the number of terms N has been shown in table 2. The exact solutions are $D_2(\lambda = 2) = 1.2494$ and $D_2(\lambda = 4) = -1.0686$, cf. table 1. Obviously, the rate of convergence is very slow.

2.16 June 28, 1995

Duration: 4h

PROBLEM 1



The figure shows a plane frame structure made up of the vertical sub-beams AB and BC and the horizontal beam CD . The beams have the length a , the constant bending stiffness EI and are all assumed to be massless and free of damping. At the points A and C the structure is simply supported. At point D a point mass m and a linear viscous damping element are attached. The damping element is acting in the vertical direction.

The structure is at rest at the time $t = 0$, where it is affected by a horizontal dynamic force $f(t)$ at point B given as

$$f(t) = \begin{cases} f_0 \frac{t}{\Delta t} & , \quad t \in [0, \Delta t] \\ 0 & , \quad t \in]\Delta t, \infty[\end{cases}$$

where f_0 and Δt are positive constants defined in the figure. The sub-beams are all assumed to be infinitely stiff against axial deformations, and only small vibrations from the static equilibrium state are considered.

Question 1 (20%. $\mu = 14.4\%$)

Formulate the equation of motion of point D .

Help:

A horizontal unit force at point B causes a vertical displacement at point D equal to $\frac{1}{4} \frac{a^3}{EI}$, and a vertical unit force at point D causes a vertical displacement at point D of magnitude $\frac{a^3}{EI}$.

Question 2 (10%. $\mu = 5.7\%$)

Solve the equation of motion formulated in question 1.

Help:

$$\int_0^t h(t-\tau)f(\tau)d\tau = \frac{f_0}{\Delta t \omega_0^2} \left(\frac{\min(t, \Delta t)}{m} + h(t - \min(t, \Delta t)) - h(t) + 2\zeta \omega_0 (H(t - \min(t, \Delta t)) - H(t)) \right)$$

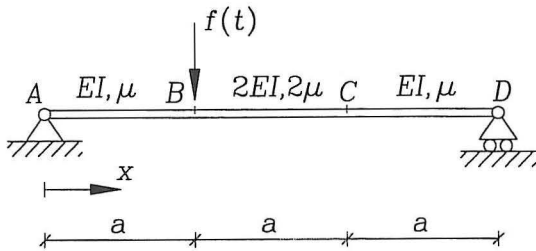
where $f(t)$ is the load shown in the figure, and $h(t)$ and $H(t)$ are given as

$$h(t) = \begin{cases} \frac{1}{m\omega_d} e^{-\zeta\omega_0 t} \sin(\omega_d t) & , \quad t \geq 0 \\ 0 & , \quad t < 0 \end{cases}$$

$$H(t) = \int_0^t h(\tau)d\tau = \frac{1}{m\omega_d\omega_0^2} \left(\omega_d - e^{-\zeta\omega_0 t} (\zeta\omega_0 \sin(\omega_d t) + \omega_d \cos(\omega_d t)) \right)$$

$$\omega_d = \omega_0 \sqrt{1 - \zeta^2}$$

PROBLEM 2



The figure shows a horizontal plane Bernoulli-Euler beam $ABCD$. The sub-beams AB and CD both have the length a , the constant bending stiffness EI and the constant mass per unit length μ . The sub-beam BC also has the length a . However, the bending stiffness is $2EI$, and the mass per unit length is 2μ . The beam is simply supported at the points A and D .

The beam is loaded by a vertical dynamic load $f(t)$ at point B . The beam is assumed to be infinitely stiff against axial deformations, and only small vertical vibrations from the static equilibrium state are considered.

Question 1 (20%. $\mu = 3.3\%$)

Determine a 2 degrees-of-freedom model for the system, as defined by the mass matrix, the stiffness matrix and the load vector, by the use of the following shape functions

$$\Phi^{(1)}(x) = \sin\left(\pi \frac{x}{3a}\right)$$

$$\Phi^{(2)}(x) = \sin\left(2\pi \frac{x}{3a}\right)$$

where x denotes a coordinate measured from point A along the beam.

Help:

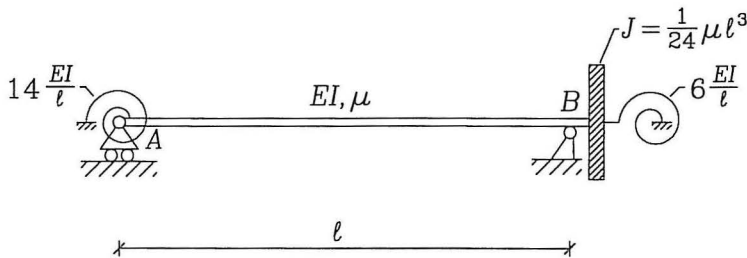
$$\int_a^b \sin^2(k\pi\xi) d\xi = \frac{b-a}{2} - \frac{\sin(2k\pi b) - \sin(2k\pi a)}{4k\pi}$$

where a and b are real constants and k is an integer.

Question 2 (15%. $\mu = 5.2\%$)

Determine the undamped circular eigenfrequencies and the undamped mode shapes of the 2 degrees-of-freedom system formulated in question 1.

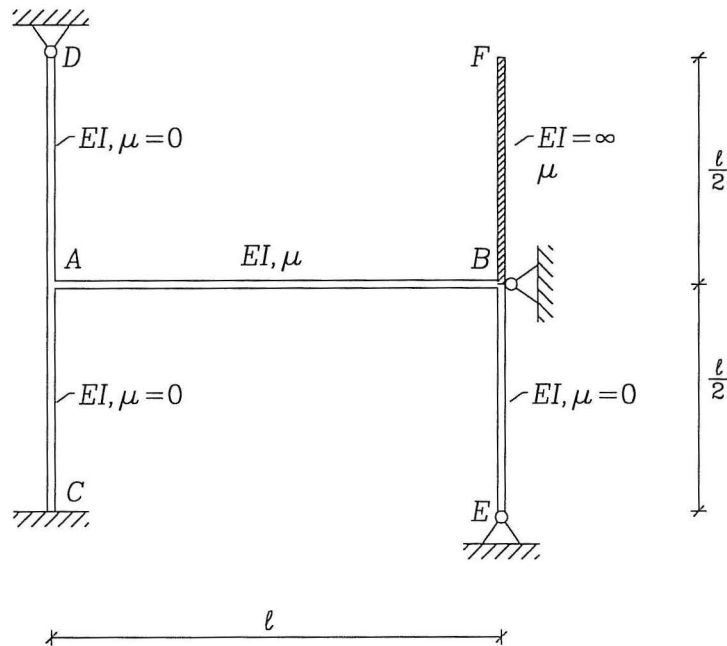
PROBLEM 3



The figure shows a horizontal, plane, linear elastic Bernoulli-Euler beam of the length ℓ . The beam is assumed to be free of damping and has the constant bending stiffness EI and the constant mass per unit length μ . The beam is supported against vertical displacements at the points A and B . At point A a linear elastic rotational spring with the spring constant $14 \frac{EI}{\ell}$ is attached. At point B a distributed mass is placed with the mass moment of inertia $J = \frac{1}{24} \mu \ell^3$ around the neutral axis of the beam AB . Besides, a linear elastic rotational spring with the spring constant $6 \frac{EI}{\ell}$ is attached. The beam is assumed to be infinitely stiff against axial deformations, and only small vertical vibrations from the static equilibrium state are considered. The influence on the dynamic response from possible axial forces is ignored.

Question 1 (20%. $\mu = 13.2\%$)

Formulate the frequency condition for the determination of the undamped circular eigenfrequencies of the beam. Numerical solution of the frequency condition is not required.



Next, consider the plane frame structure shown in the figure, made up of the linear elastic Bernoulli-Euler beams AB , AC , AD , BE and the infinitely stiff beam BF . Beam AB has the length l , the constant bending stiffness EI and the constant mass per unit length μ . The beams AC , AD and BE are all assumed to be massless ($\mu = 0$), and all have the same length $\frac{l}{2}$ and the same constant bending stiffness EI . The infinitely stiff beam BF ($EI = \infty$) also has the length $\frac{l}{2}$ and the constant mass per unit length μ . The structure is fixed at point C and simply supported at the points B , D and E . All beams are assumed to be infinitely stiff against axial deformations, and only small vibrations of the beam elements in the transverse direction from the static equilibrium state are considered.

Question 2 (15%. $\mu = 8.5\%$)

Formulate the frequency condition for the determination of the undamped circular eigenfrequencies of the beam.

PROBLEM 1

Question 1:

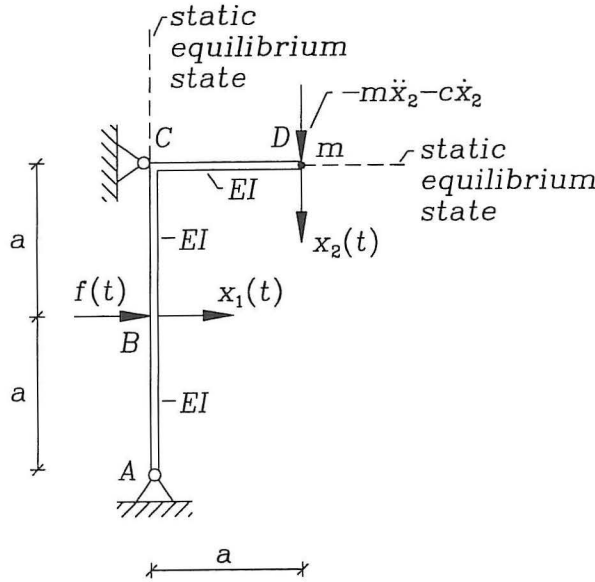


Fig. 1: Forces on the free structure.

The frame $ABCD$ is massless and infinitely stiff against axial elongations. Hence, the system has but a single degree of freedom which is selected as the vertical displacement $x_2(t)$ of the point mass from the statical equilibrium state with sign as defined in fig. 1. Further, an artificial degree of freedom $x_1(t)$ is introduced indicating the horizontal displacement from the static equilibrium state of the point B , where the indirectly acting force $f(t)$ is applied, cf. (2-160). The beam is cut free from the damper, and the damper force $c\dot{x}_2(t)$ is applied as an external load with sign as defined in fig. 1. Further, the inertial load $-m\ddot{x}_2(t)$ is applied as an external dynamic load acting on the mass according to d'Alembert's principle. The equation of motion for the mass reads

$$x_2(t) = \delta_{21}f(t) + \delta_{22}(-m\ddot{x}_2 - c\dot{x}_2) \quad (1)$$

The flexibility coefficients are given as

$$\delta_{21} = -\frac{1}{4} \frac{a^3}{EI}, \quad \delta_{22} = \frac{a^2}{EI} \quad (2)$$

Insertion (2) into (1) provides the following equation of motion for the vertical displacements of point D

$$\left. \begin{aligned} m(\ddot{x}_2 + 2\zeta\omega_0\dot{x}_2 + \omega_0^2x_2) &= -\frac{1}{4}f(t) \\ x_2(0) = \dot{x}_2(0) &= 0 \end{aligned} \right\} \quad (3)$$

where the circular eigenfrequency ω_0 and the damping ratio ζ are given as

$$\omega_0 = \sqrt{\frac{1}{m\delta_{22}}} = \sqrt{\frac{EI}{ma^3}} \quad (4)$$

$$\zeta = \frac{c}{2\omega_0 m} = c\sqrt{\frac{a^3}{4EI m}} \quad (5)$$

The initial values $x_2(0) = \dot{x}_2(0) = 0$ shown in (3) follow, since the structure is at rest at the statical equilibrium state at the time $t = 0$.

Question 2:

Because of the initial conditions $x_2(0) = \dot{x}_2(0) = 0$, the response is given by Duhamel's integral, cf. (2-121)

$$x_2(t) = \int_0^t h(t-\tau) \left(-\frac{1}{4}f(\tau)\right) d\tau \quad (6)$$

where $h(t)$ is the impulse response function given by (2-110)

$$h(t) = \begin{cases} \frac{1}{m\omega_d} e^{-\zeta\omega_0 t} \sin(\omega_d t) & , \quad t \geq 0 \\ 0 & , \quad t < 0 \end{cases} \quad (7)$$

From the results allocated to the problem it then follows that

$$x_2(t) = -\frac{1}{4} \frac{f_0}{\Delta t \omega_0^2} \cdot \left(\frac{\min(t, \Delta t)}{m} + h(t - \min(t, \Delta t)) - h(t) + 2\zeta\omega_0 \left(H(t - \min(t, \Delta t)) - H(t) \right) \right) \quad (8)$$

where

$$H(t) = \int_0^t h(\tau) d\tau = \frac{1}{m\omega_d \omega_0^2} \left(\omega_d - e^{-\zeta\omega_0 t} (\zeta\omega_0 \sin(\omega_d t) + \omega_d \cos(\omega_d t)) \right) \quad (9)$$

$$\omega_d = \omega_0 \sqrt{1 - \zeta^2} \quad (10)$$

ω_d is the damped circular eigenfrequency, cf. (2-49).

PROBLEM 2

Question 1:

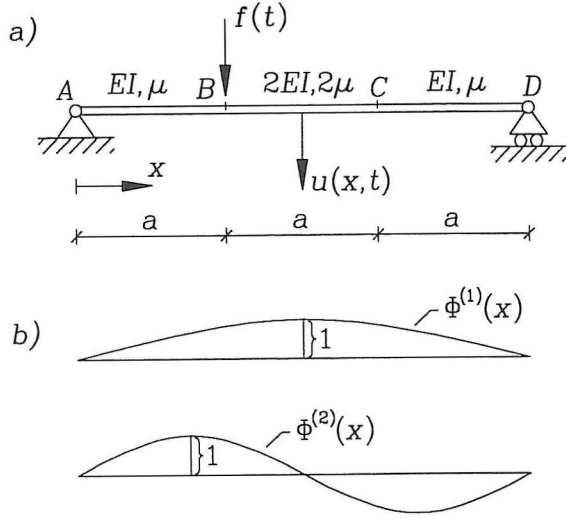


Fig. 1: a) Structural system. b) Sinusoidal shape functions.

For the structural system shown in fig. 1a the sinusoidal shape functions shown in fig. 1b will be used. This means that the displacement field $u(x, t)$ is assumed on the form

$$u(x, t) \simeq x_1(t)\Phi^{(1)}(x) + x_2(t)\Phi^{(2)}(x) \quad (1)$$

The following equation of motion is obtained for the generalized coordinates $x_1(t)$, $x_2(t)$, cf. (5-59), (5-60), (5-61)

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \quad (2)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \quad (3)$$

$$\begin{aligned} M_{11} &= \int_0^{3a} \mu(x) (\Phi^{(1)}(x))^2 dx = \int_0^{3a} \mu(x) \sin^2\left(\pi \frac{x}{3a}\right) dx = \\ &= 3a\mu \left(\int_0^{\frac{1}{3}} \sin^2(\pi\xi) d\xi + 2 \int_{\frac{1}{3}}^{\frac{2}{3}} \sin^2(\pi\xi) d\xi + \int_{\frac{2}{3}}^1 \sin^2(\pi\xi) d\xi \right) = \\ &= 3a\mu \left(\frac{1}{6} - \frac{\sin \frac{2\pi}{3}}{4\pi} + 2 \left(\frac{1}{6} - \frac{\sin \frac{4\pi}{3} - \sin \frac{2\pi}{3}}{4\pi} \right) + \frac{1}{6} - \frac{0 - \sin \frac{4\pi}{3}}{4\pi} \right) = 3a\mu \left(\frac{2}{3} + \frac{\sqrt{3}}{4\pi} \right) \end{aligned} \quad (4a)$$

$$M_{12} = \int_0^{3a} \mu(x) \Phi^{(1)}(x) \Phi^{(2)}(x) dx = \int_0^{3a} \mu(x) \sin\left(\pi \frac{x}{3a}\right) \sin\left(2\pi \frac{x}{3a}\right) dx = 0 \quad (4b)$$

$$\begin{aligned}
M_{22} &= \int_0^{3a} \mu(x) (\Phi^{(2)}(x))^2 dx = \int_0^{3a} \mu(x) \sin^2(2\pi \frac{x}{3a}) dx = \\
&= 3a\mu \left(\int_0^{\frac{1}{3}} \sin^2(2\pi\xi) d\xi + 2 \int_{\frac{1}{3}}^{\frac{2}{3}} \sin^2(2\pi\xi) d\xi + \int_{\frac{2}{3}}^1 \sin^2(2\pi\xi) d\xi \right) = \\
&= 3a\mu \left(\frac{1}{6} - \frac{\sin \frac{4\pi}{3}}{8\pi} + 2 \left(\frac{1}{6} - \frac{\sin \frac{8\pi}{3} - \sin \frac{4\pi}{3}}{8\pi} \right) + \frac{1}{6} - \frac{0 - \sin \frac{8\pi}{3}}{8\pi} \right) = 3a\mu \left(\frac{2}{3} - \frac{\sqrt{3}}{8\pi} \right) \quad (4c)
\end{aligned}$$

$$\begin{aligned}
K_{11} &= \int_0^{3a} EI(x) \left(\frac{d^2 \Phi^{(1)}(x)}{dx^2} \right)^2 dx = \left(\frac{\pi}{3a} \right)^4 \int_0^{3a} EI(x) \sin^2 \left(\pi \frac{x}{3a} \right) dx = \\
&= \left(\frac{\pi}{3a} \right)^4 3aEI \left(\int_0^{\frac{1}{3}} \sin^2(\pi\xi) d\xi + 2 \int_{\frac{1}{3}}^{\frac{2}{3}} \sin^2(\pi\xi) d\xi + \int_{\frac{2}{3}}^1 \sin^2(\pi\xi) d\xi \right) = \\
&= \left(\frac{\pi}{3a} \right)^4 3aEI \left(\frac{2}{3} + \frac{\sqrt{3}}{4\pi} \right) \quad (5a)
\end{aligned}$$

$$\begin{aligned}
K_{12} &= \int_0^{3a} EI(x) \frac{d^2 \Phi^{(1)}(x)}{dx^2} \frac{d^2 \Phi^{(2)}(x)}{dx^2} dx = \\
&= \left(\frac{\pi}{3a} \right)^2 \left(\frac{2\pi}{3a} \right)^2 \int_0^{3a} EI(x) \sin \left(\pi \frac{x}{3a} \right) \sin \left(2\pi \frac{x}{3a} \right) dx = 0 \quad (5b)
\end{aligned}$$

$$\begin{aligned}
K_{22} &= \int_0^{3a} EI(x) \left(\frac{d^2 \Phi^{(2)}(x)}{dx^2} \right)^2 dx = \left(\frac{2\pi}{3a} \right)^4 \int_0^{3a} EI(x) \sin^2 \left(2\pi \frac{x}{3a} \right) dx = \\
&= \left(\frac{2\pi}{3a} \right)^4 3aEI \left(\int_0^{\frac{1}{3}} \sin^2(2\pi\xi) d\xi + 2 \int_{\frac{1}{3}}^{\frac{2}{3}} \sin^2(2\pi\xi) d\xi + \int_{\frac{2}{3}}^1 \sin^2(2\pi\xi) d\xi \right) = \\
&= \left(\frac{2\pi}{3a} \right)^4 3aEI \left(\frac{2}{3} - \frac{\sqrt{3}}{8\pi} \right) \quad (5c)
\end{aligned}$$

(4b) and (5b) follow since $\sin(\pi \frac{x}{3a})$, $\mu(x)$, $EI(x)$ are symmetric, and $\sin(2\pi \frac{x}{3a})$ is anti-symmetric around the midpoint of the beam at $x = \frac{3a}{2}$. The evaluation of (5a) and (5c) follows from the corresponding results (4a) and (4c), because $\mu(x)$ and $EI(x)$ are identically distributed, i.e. $\mu(x)$ is proportional to $EI(x)$.

The concentrated load at $B(x = a)$ can formally be written as the following dynamic load per unit length, cf. (4-68)

$$f_d(x, t) = f(t)\delta(x - a) \quad (6)$$

The load vector components then become, cf. (5-44)

$$f_1(t) = \int_0^{3a} \Phi^{(1)}(x) f_d(x, t) dx = \int_0^{3a} \sin \left(\pi \frac{x}{3a} \right) f(t)\delta(x - a) dx = \sin \frac{\pi}{3} f(t) = \frac{\sqrt{3}}{2} f(t) \quad (7a)$$

$$f_2(t) = \int_a^{3a} \Phi^{(2)}(x) f_d(x, t) dx = \int_0^{3a} \sin\left(2\pi \frac{x}{3a}\right) f(t) \delta(x-a) dx = \sin \frac{2\pi}{3} f(t) = \frac{\sqrt{3}}{2} f(t) \quad (7b)$$

The results can be assembled into the following matrix results

$$\mathbf{M} = 3a\mu \begin{bmatrix} \frac{2}{3} + \frac{\sqrt{3}}{4\pi} & 0 \\ 0 & \frac{2}{3} - \frac{\sqrt{3}}{8\pi} \end{bmatrix}, \quad \mathbf{K} = \left(\frac{\pi}{3a}\right)^4 3aEI \begin{bmatrix} \frac{2}{3} + \frac{\sqrt{3}}{4\pi} & 0 \\ 0 & 2^4 \left(\frac{2}{3} - \frac{\sqrt{3}}{8\pi}\right) \end{bmatrix}$$

$$\mathbf{f}(t) = \frac{\sqrt{3}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} f(t) \quad (8)$$

Question 2:

The circular eigenfrequencies ω_j and the eigenmodes $\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix}$ are obtained as non-trivial solutions to the homogeneous linear equations, cf. (3-42)

$$\begin{bmatrix} \omega^2 - \omega_0^2 & 0 \\ 0 & \omega^2 - 2^4 \omega_0^2 \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (9)$$

where

$$\omega_0^2 = \left(\frac{\pi}{3a}\right)^4 \frac{EI}{\mu} \quad (10)$$

The circular eigenfrequencies and the eigenmodes become

$$\omega_j^2 = j^4 \omega_0^2 = \left(\frac{j\pi}{3a}\right)^4 \frac{EI}{\mu}, \quad j = 1, 2 \quad (11)$$

$$\Phi^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Phi^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (12)$$

As seen the mass matrix and the stiffness matrix are diagonal. Hence, the selected generalized coordinates are also the undamped modal coordinates of the discretized system.

The circular frequencies (11) are identical to those of a simply supported beam of the length $3a$ and with constant bending stiffness EI and mass per unit length μ , cf. (4-33). This is because the eigenfrequencies are mainly depending on the fraction $\frac{EI(x)}{\mu(x)}$ which is constant, rather than on the absolute values of $EI(x)$ and $\mu(x)$. Notice that this result is only approximately true. The exact solution for the undamped circular eigenfrequencies to the considered system will be slightly different from the result (11).

PROBLEM 3

Question 1:

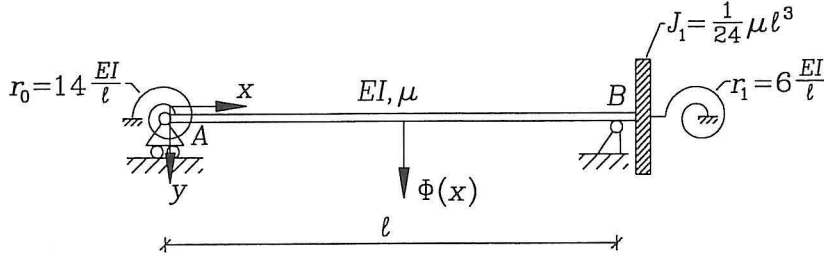


Fig. 1: Beam with constant bending stiffness and mass per unit length.

A local (x, y) -coordinate system is defined for the beam element as shown in fig. 1. The beam has constant bending stiffness EI , constant mass per unit length μ , and the axial force in the statical equilibrium state is $N = 0$. Hence, the eigenmodes are given by (4-18), (4-19).

$$\Phi(x) = A \sin\left(\lambda \frac{x}{l}\right) + B \cos\left(\lambda \frac{x}{l}\right) + C \sinh\left(\lambda \frac{x}{l}\right) + D \cosh\left(\lambda \frac{x}{l}\right) \quad (1)$$

$$\lambda^4 = \frac{\mu \omega^2 l^4}{EI} \quad (2)$$

The boundary conditions at point $A(x = 0)$ and point $B(x = l)$ become, see (4-13)

$$\Phi(0) = 0 \quad (3)$$

$$EI \frac{d^2 \Phi(0)}{dx^2} = r_0 \frac{d\Phi(0)}{dx} \Rightarrow \frac{d^2 \Phi(0)}{dx^2} = \frac{14}{l} \frac{d\Phi(0)}{dx} \quad (4)$$

$$\Phi(l) = 0 \quad (5)$$

$$EI \frac{d^2 \Phi(l)}{dx^2} = -(r_1 - \omega^2 J_1) \frac{d\Phi(l)}{dx} \Rightarrow \frac{d^2 \Phi(l)}{dx^2} = -\frac{1}{l} \left(6 - \frac{1}{24} \lambda^4\right) \frac{d\Phi(l)}{dx} \quad (6)$$

The geometrical boundary conditions (3) and (5) state that the displacement is zero at all times at $x = 0$ and $x = l$. The mechanical boundary conditions (4) and (6) state that the bending moment at point A must balance the moment in the rotational spring and the bending moment at point B must balance the moment in the rotational spring and the d'Alembert moment on the distributed mass. In the last statement of (6) ω^2 has been eliminated in favour of λ^4 by (2).

(3) implies

$$B + D = 0 \quad (7)$$

Then (1) reduces to

$$\Phi(x) = A \sin\left(\lambda \frac{x}{l}\right) + B \left(\cos\left(\lambda \frac{x}{l}\right) - \cosh\left(\lambda \frac{x}{l}\right)\right) + C \sinh\left(\lambda \frac{x}{l}\right) \quad (8)$$

Inserting (4), (5) and (6) into (8) the following homogeneous equations are obtained for the determination of A , B and C

$$\left. \begin{aligned} -2 \frac{\lambda^2}{l^2} B &= \frac{14}{l} \frac{\lambda}{l} (A + C) \\ \sin \lambda A + (\cos \lambda - \cosh \lambda) B + \sinh \lambda C &= 0 \\ \left(\frac{\lambda}{l}\right)^2 \left(-\sin \lambda A - (\cos \lambda + \cosh \lambda) B + \sinh \lambda C\right) &= \\ -\frac{1}{l} \left(6 - \frac{1}{24} \lambda^4\right) \frac{\lambda}{l} \left(\cos \lambda A - (\sin \lambda + \sinh \lambda) B + \cosh \lambda C\right) & \end{aligned} \right\} \Rightarrow$$

$$\begin{bmatrix} K_{11}(\lambda) & K_{12}(\lambda) & K_{13}(\lambda) \\ K_{21}(\lambda) & K_{22}(\lambda) & K_{23}(\lambda) \\ K_{31}(\lambda) & K_{32}(\lambda) & K_{33}(\lambda) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (9)$$

$$K_{11}(\lambda) = 7 \quad (9a)$$

$$K_{12}(\lambda) = \lambda \quad (9b)$$

$$K_{13}(\lambda) = 7 \quad (9c)$$

$$K_{21}(\lambda) = \sin \lambda \quad (9d)$$

$$K_{22}(\lambda) = \cos \lambda - \cosh \lambda \quad (9e)$$

$$K_{23}(\lambda) = \sinh \lambda \quad (9f)$$

$$K_{31}(\lambda) = \lambda \sin \lambda - \left(6 - \frac{1}{24} \lambda^4\right) \cos \lambda \quad (9g)$$

$$K_{32}(\lambda) = \lambda (\cos \lambda + \cosh \lambda) + \left(6 - \frac{1}{24} \lambda^4\right) (\sin \lambda + \sinh \lambda) \quad (9h)$$

$$K_{33}(\lambda) = -\lambda \sinh \lambda - \left(6 - \frac{1}{24}\lambda^4\right) \cosh \lambda \quad (9i)$$

Non-trivial solutions $A \neq 0 \vee B \neq 0 \vee C \neq 0$ are obtained if the determinant of (9) is 0. This may be evaluated, and the resulting frequency condition reduced by means of trigonometric and hyperbolic identities. The 6 lowest eigenvalues become

$$\lambda_j = \begin{cases} 3.58185 & , & j = 1 \\ 4.85229 & , & j = 2 \\ 7.57677 & , & j = 3 \\ 10.60931 & , & j = 4 \\ 13.68753 & , & j = 5 \\ 16.78470 & , & j = 6 \end{cases} \quad (10)$$

The corresponding circular undamped eigenfrequencies follow from (2)

$$\omega_j = \lambda_j^2 \sqrt{\frac{EI}{\mu l^4}} = \begin{cases} 12.830 \sqrt{\frac{EI}{\mu l^4}} & , & j = 1 \\ 23.545 \sqrt{\frac{EI}{\mu l^4}} & , & j = 2 \\ 57.409 \sqrt{\frac{EI}{\mu l^4}} & , & j = 3 \\ 112.557 \sqrt{\frac{EI}{\mu l^4}} & , & j = 4 \\ 187.348 \sqrt{\frac{EI}{\mu l^4}} & , & j = 5 \\ 281.726 \sqrt{\frac{EI}{\mu l^4}} & , & j = 6 \end{cases} \quad (11)$$

Question 2:

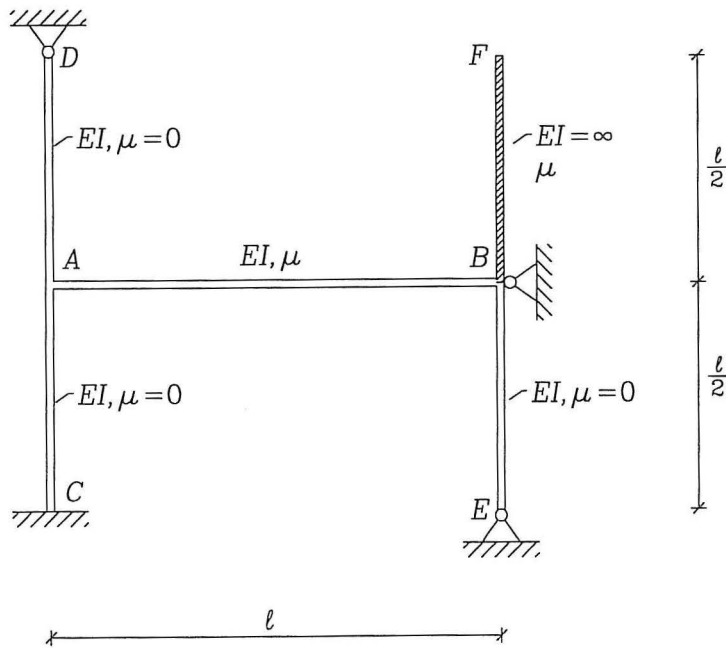


Fig. 2: Plane frame structure.

The beams AD and AC are both massless. Hence, these beams merely act as rotational springs against the vertical displacement of the beam AB . Taking the boundary conditions at the points C and D into consideration the equivalent rotational spring constants of the beams become $r_{AD} = 3 \frac{EI}{l/2} = 6 \frac{EI}{l}$ and $r_{AC} = 4 \frac{EI}{l/2} = 8 \frac{EI}{l}$, respectively. The effect of both beams is then equivalent to that of a linear elastic rotational spring with the constant $r_0 = 6 \frac{EI}{l} + 8 \frac{EI}{l} = 14 \frac{EI}{l}$.

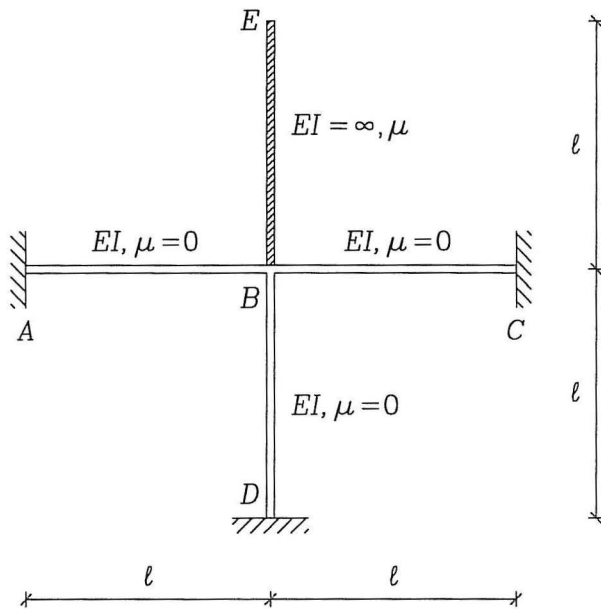
Beam BE is massless and is then equivalent to a linear elastic rotational spring with the spring constant $r_1 = 3 \frac{EI}{l/2} = 6 \frac{EI}{l}$. Beam BF is infinitely stiff. The dynamic influence on the vertical vibrations of beam AB is then equivalent to that of a distributed mass with mass moment of inertia of $J_1 = \frac{1}{3} \mu \left(\frac{l}{2} \right)^3 = \frac{1}{24} \mu l^3$.

Hence, the eigenfrequencies of the structure can be analysed by the equivalent system described in question 1, and the lowest 6 eigenfrequencies are given by (11).

2.17 June 21, 1996

Duration: 4h

PROBLEM 1

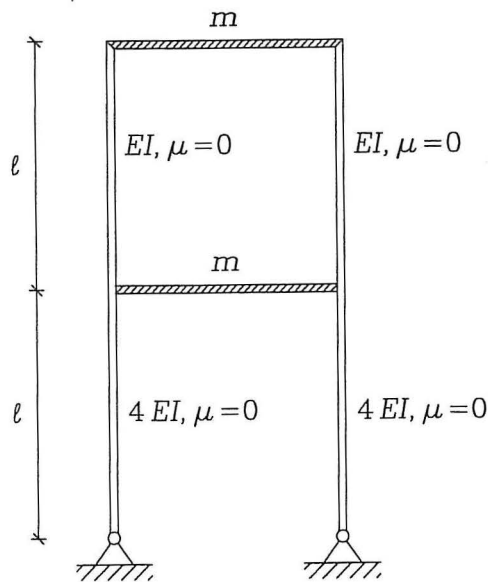


The figure shows a plane frame structure made up of the horizontal sub-beams AB and BC and the vertical sub-beams BC and BD . All sub-beams have the length l . Beams AB , BC and BD are massless Bernoulli-Euler beams with the constant bending stiffness EI . The structure is fixed at the points A , C and D . Beam BE is infinitely stiff and has the mass per unit length μ . The sub-beams are all assumed to be infinitely stiff against axial deformations, and only small vibrations in the plane of the structure are considered.

Question 1 (15%. $\mu = 6.6\%$)

Determine the circular eigenfrequency of the structure.

PROBLEM 2



The figure shows a plane two-storey frame structure. Both of the storey beams have the total mass m , and are assumed to be infinitely stiff against axial as well as bending deformations. The columns are massless Bernoulli-Euler beams of the length l , and are assumed to be infinitely stiff against axial deformations. The columns between the 1st and 2nd storey beam have the constant bending stiffness EI , and the columns between the support points and the 1st storey beam have the constant bending stiffness $4EI$. The lower columns are simply supported at the ground surface. Only small vibrations in the plane of the structure are considered.

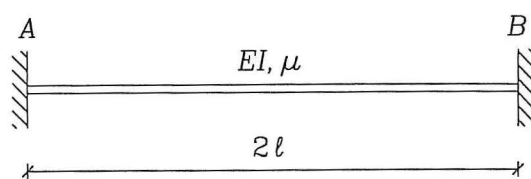
Question 1 (20%. $\mu = 14.8\%$)

Determine the undamped circular eigenfrequencies and eigenmodes of the structure.

Question 2 (15%. $\mu = 7.2\%$)

The structure is assumed to be linear viscous damped. Calculate the damping matrix \mathbf{C} , when the modal damping ratios have been measured to $\zeta_1 = 1\%$ and $\zeta_2 = 2\%$, and modal decoupling can be assumed, argued by the well separated circular eigenfrequencies.

PROBLEM 3

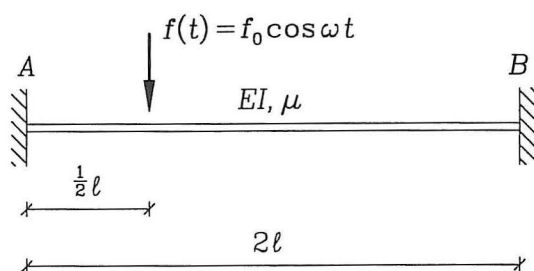


The figure shows a horizontal, plane Bernoulli-Euler beam of the length l , free of damping and with the constant bending stiffness EI and the constant mass per unit length μ . The beam is fixed at the points A and B and is assumed to be infinitely stiff against axial deformations. Only small vertical vibrations from the static equilibrium state is considered, and the influence from a possible axial force on the response is ignored.

Question (15%. $\mu = 11.0\%$)

Formulate the frequency condition for the determination of the undamped circular eigenfrequencies of the beam. Numerical solution of the frequency condition is not required.

PROBLEM 4



The figure shows the same plane Bernoulli-Euler beam as considered in problem 3. However, now the beam is loaded by a vertical harmonically varying force with the amplitude f_0 and the circular frequency ω , acting at the quarter point from point A . The bending stiffness, the mass per unit length and the length of the beam, as well as the assumptions listed in problem 3 are still valid.

Question 1 (20%. $\mu = 2.1\%$)

Formulate a finite element model of the beam using consistent mass matrices, based on a division of the beam in 2 sub-beam elements. The global mass matrix, the global stiffness matrix and the global load vector are requested. Next, calculate the undamped circular eigenfrequencies and eigenmodes of the discrete model and make a sketch of the eigenmodes.

Question 2 (15%. $\mu = 3.8\%$)

Determine the stationary response of the discretized system from the harmonically varying force, as the response from possible initial values has died away.

SOLUTIONS

PROBLEM 1

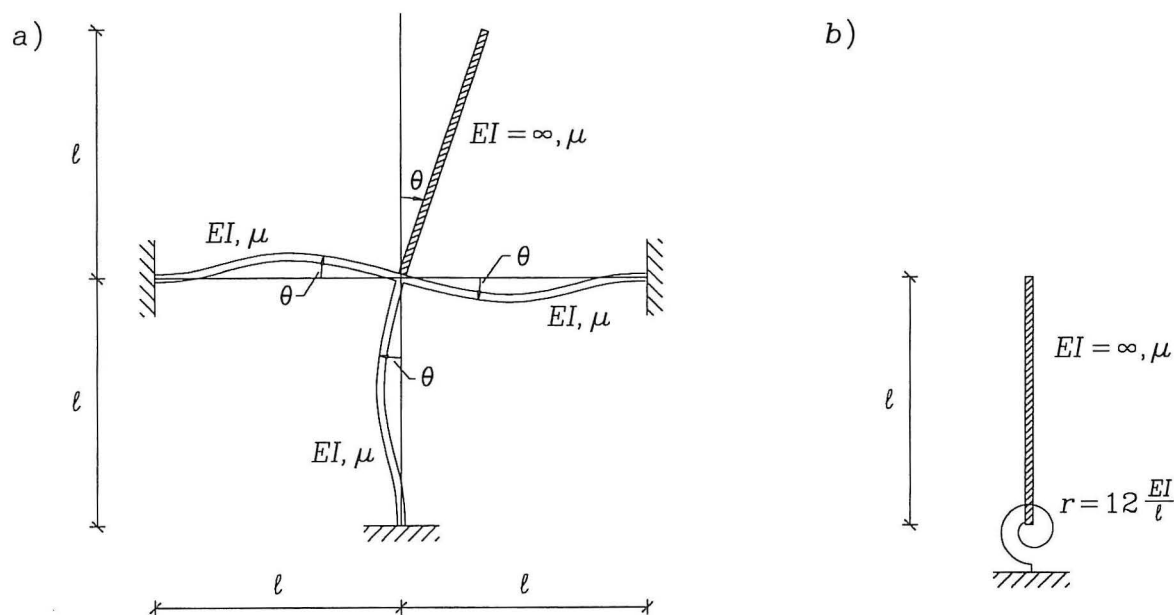


Fig. 1: a) Definition of degree of freedom. b) Equivalent system.

The system has a single degree of freedom, which is selected as the rotation $\theta(t)$ of point B with the sign as defined in fig. 1a. Because the beams AB , BC and DB are massless they merely act as linear elastic rotational springs with the spring constant $4 \frac{EI}{\ell}$. The system may then be analysed by the equivalent system shown in fig. 1b with the rotational spring constant given by

$$r = 3 \cdot 4 \frac{EI}{\ell} = 12 \frac{EI}{\ell} \quad (1)$$

The mass moment of inertia of beam BE around point B is given as

$$J = \frac{1}{3} \mu \ell^3 \quad (2)$$

The equation of motion then reads

$$\begin{aligned} J \ddot{\theta} + r \theta &= 0 \quad \Rightarrow \\ \ddot{\theta} + \omega_0^2 \theta &= 0 \end{aligned}$$

where the circular eigenfrequency ω_0 is given as, cf. (2-7)

$$\omega_0 = \sqrt{\frac{r}{J}} = 6 \sqrt{\frac{EI}{\mu \ell^4}} \quad (4)$$

PROBLEM 2

Question 1:

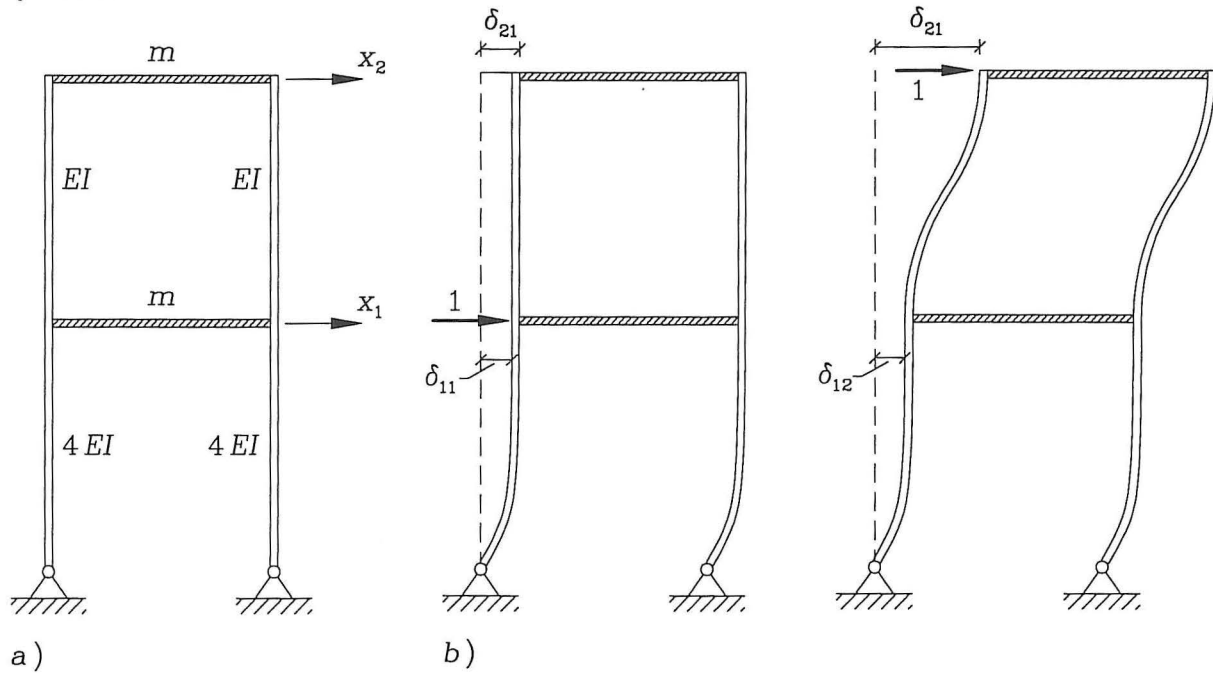


Fig. 1: Two-storey frame structure. a) Definition of degrees of freedom. b) Deformation of frame from unit forces at x_1 and x_2 .

The structure has two degrees of freedom $x_1(t)$ and $x_2(t)$, which are selected as the horizontal displacements of the 1st and 2nd storey beams from the static equilibrium state, see fig. 1a. The equations of motion then read, cf. (3-11)

$$\begin{cases} x_1(t) = \delta_{11}(-m\ddot{x}_1) + \delta_{12}(-m\ddot{x}_2) \\ x_2(t) = \delta_{21}(-m\ddot{x}_1) + \delta_{22}(-m\ddot{x}_2) \end{cases} \quad (1)$$

The flexibility coefficients and the flexibility matrix become, cf. Example 3-1 in the textbook

$$\begin{aligned} \left. \begin{aligned} \delta_{11} &= \delta_{12} = \delta_{21} = \frac{l^3}{6 \cdot 4EI} = \frac{1}{24} \frac{l^3}{EI} \\ \delta_{22} &= \frac{l^3}{24EI} + \frac{l^3}{6 \cdot 4EI} = \frac{2}{24} \frac{l^3}{EI} \end{aligned} \right\} \Rightarrow \\ \mathbf{D} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \frac{1}{24} \frac{l^3}{EI} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \end{aligned} \quad (2)$$

The equations of motion can then be written as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \quad (3)$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{K} = \mathbf{D}^{-1} = 24 \frac{EI}{l^3} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad (4)$$

Undamped circular eigenfrequencies and eigenmodes can then be determined from the eigenvalue problem, cf. (3-42)

$$\begin{bmatrix} 2 - \lambda_j & -1 \\ -1 & 1 - \lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \lambda_j = \frac{1}{24} \frac{m\omega_j^2 l^3}{EI}, \quad j = 1, 2 \quad (5)$$

The characteristic equation becomes

$$\lambda_j^2 - 3\lambda_j + 1 = 0 \Rightarrow$$

$$\lambda_j = \begin{cases} \frac{1}{2}(3 - \sqrt{5}) & , \quad j = 1 \\ \frac{1}{2}(3 + \sqrt{5}) & , \quad j = 2 \end{cases} \quad (6)$$

$$\omega_j = \begin{cases} \sqrt{12(3 - \sqrt{5})} \sqrt{\frac{EI}{ml^3}} & , \quad j = 1 \\ \sqrt{12(3 + \sqrt{5})} \sqrt{\frac{EI}{ml^3}} & , \quad j = 2 \end{cases} \quad (7)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix} \quad (8)$$

The component at the 1st storey $\Phi_1^{(j)}$ is determined from the 2nd equation in (5)

$$\Phi_1^{(j)} = 1 - \lambda_j = \begin{cases} \frac{1}{2}(\sqrt{5} - 1) & , \quad j = 1 \\ \frac{1}{2}(\sqrt{5} + 1) & , \quad j = 2 \end{cases} \quad (9)$$

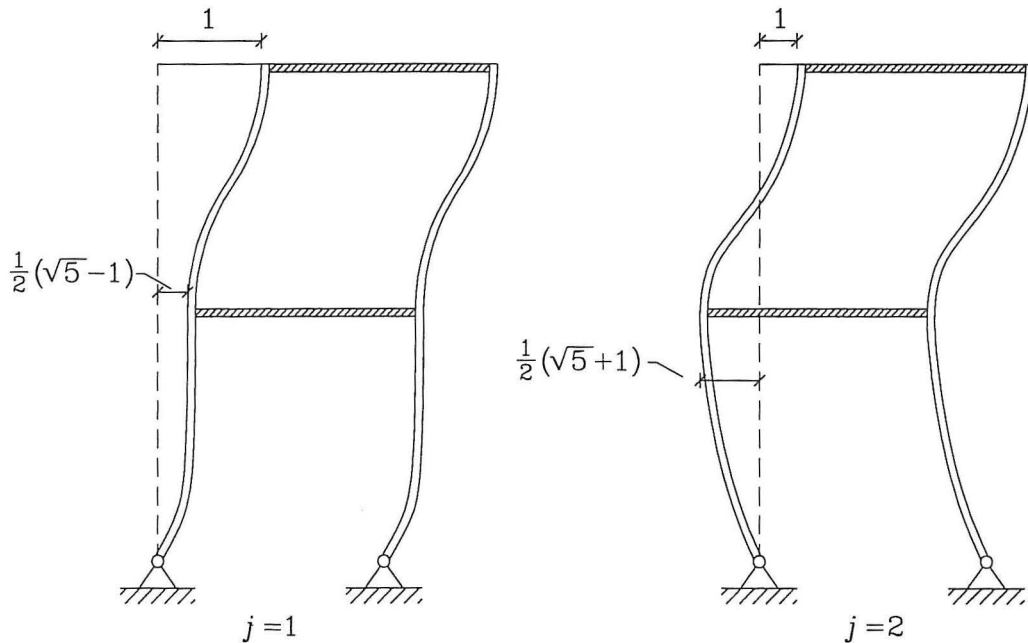


Fig. 2: Eigenmodes of two-storey frame structure.

Question 2:

The damping matrix can be represented by the following Rayleigh damping model, cf. (3-284), (4)

$$\mathbf{C} = a_0 \mathbf{M} + a_1 \mathbf{K} = a_0 m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 24 \frac{a_1 EI}{l^3} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad (10)$$

where the expansion coefficients a_0 and a_1 are calibrated from the linear equations, cf. (3-287), (7)

$$\begin{aligned} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} &= \frac{2\omega_1\omega_2}{\omega_2^2 - \omega_1^2} \begin{bmatrix} \omega_2 & -\omega_1 \\ -\frac{1}{\omega_2} & \frac{1}{\omega_1} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \frac{4}{5} \sqrt{15(3 + \sqrt{5})} \begin{bmatrix} \left(\zeta_1 - \frac{1}{2}(3 - \sqrt{5})\zeta_2 \right) \sqrt{\frac{EI}{ml^3}} \\ \left(\zeta_2 - \frac{1}{2}(3 - \sqrt{5})\zeta_1 \right) \sqrt{\frac{ml^3}{576EI}} \end{bmatrix} \Rightarrow \\ \mathbf{C} &= \frac{4}{5} \sqrt{15(3 + \sqrt{5})} \left(\left(\zeta_1 - \frac{1}{2}(3 - \sqrt{5})\zeta_2 \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\zeta_2 - \frac{1}{2}(3 - \sqrt{5})\zeta_1 \right) \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \right) \sqrt{\frac{mEI}{l^3}} \quad (11) \end{aligned}$$

PROBLEM 3

The beam has constant bending stiffness EI , constant mass per unit length μ and the axial force in the static equilibrium state $N = 0$. Hence, the eigenmode is given by (4-18), (4-19)

$$\Phi(x) = A \sin\left(\lambda \frac{x}{2l}\right) + B \cos\left(\lambda \frac{x}{2l}\right) + C \sinh\left(\lambda \frac{x}{2l}\right) + D \cosh\left(\lambda \frac{x}{2l}\right) \quad (1)$$

$$\lambda^4 = \frac{\mu\omega^2(2l)^4}{EI} \quad (2)$$

where x is a coordinate along the beam measured from A . The boundary conditions (all geometrical) become

$$\Phi(0) = \frac{d}{dx}\Phi(0) = \Phi(2l) = \frac{d}{dx}\Phi(2l) = 0 \quad (3)$$

The boundary conditions at $x = 0$ provide $D = -B$ and $C = -A$, cf. (4-37), leading to the following reduced expression for the eigenmode

$$\Phi(x) = A \left(\sin\left(\lambda \frac{x}{2l}\right) - \sinh\left(\lambda \frac{x}{2l}\right) \right) + B \left(\cos\left(\lambda \frac{x}{2l}\right) - \cosh\left(\lambda \frac{x}{2l}\right) \right) \quad (4)$$

Next, (4) is inserted into the boundary conditions at $x = 2l$, leading the following homogenous linear equations

$$\begin{bmatrix} \sin \lambda - \sinh \lambda & \cos \lambda - \cosh \lambda \\ \cos \lambda - \cosh \lambda & -\sin \lambda - \sinh \lambda \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (5)$$

The frequency condition then becomes

$$\begin{aligned}
 & (\sin \lambda - \sinh \lambda)(-\sin \lambda - \sinh \lambda) - (\cos \lambda - \cosh \lambda)(\cos \lambda + \cosh \lambda) = 0 \Rightarrow \\
 & -\sin^2 \lambda + \sinh^2 \lambda - \cos^2 \lambda - \cosh^2 \lambda + 2 \cos \lambda \cosh \lambda = 0 \Rightarrow \\
 & \cos \lambda \cosh \lambda - 1 = 0
 \end{aligned} \tag{6}$$

The two lowest solutions of (6) and the corresponding circular eigenfrequencies as obtained from (2) become

$$\begin{aligned}
 \lambda_j &= \begin{cases} 4.7300407 & , \quad j = 1 \\ 7.8532046 & , \quad j = 2 \end{cases} \Rightarrow \\
 \omega_j &= \begin{cases} 5.593321 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 1 \\ 15.418206 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 2 \end{cases}
 \end{aligned} \tag{7}$$

PROBLEM 4

Question 1:

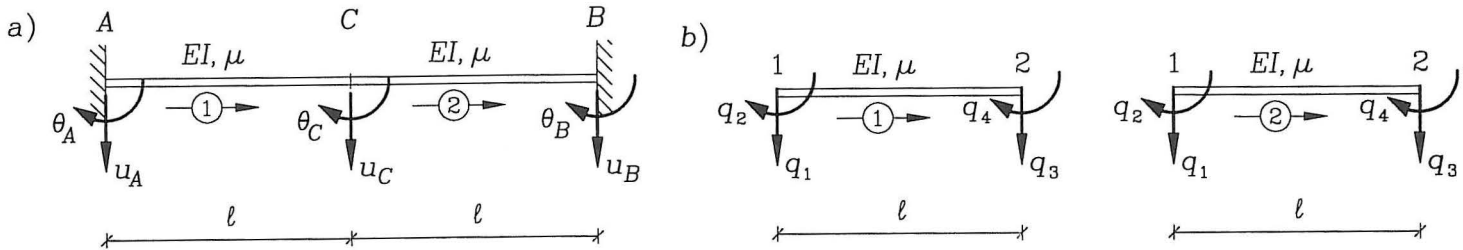


Fig. 1: Division of beam into two sub-beams. a) Definition of structural data and global degrees of freedom. b) Definition of local degrees of freedom.

An artificial node is introduced at the middle of the beam. Global degrees of freedom for the two sub-beams have been defined in fig. 1a. Global and local coordinate systems are co-directional. The element stiffness and consistent mass matrices in global coordinates are identical for both elements. Below, these as well as the element load vectors have been indicated

$$\mathbf{K}_j = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{j,11} & \mathbf{K}_{j,12} \\ \mathbf{K}_{j,12}^T & \mathbf{K}_{j,22} \end{bmatrix}, \quad j = 1, 2 \tag{1}$$

$$\mathbf{M}_j = \frac{\mu l}{420} \begin{bmatrix} 156 & 22l & | & 54 & -13l \\ 22l & 4l^2 & | & 13l & -3l^2 \\ \hline 54 & 13l & | & 156 & -22l \\ -13l & -3l^2 & | & -22l & 4l^2 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{j,11} & | & \mathbf{M}_{j,12} \\ \hline \mathbf{M}_{j,12}^T & | & \mathbf{M}_{j,22} \end{bmatrix}, \quad j = 1, 2 \quad (2)$$

$$\mathbf{F}_1(t) = \int_0^l \begin{bmatrix} N_2(x) \\ N_3(x) \\ \hline N_5(x) \\ N_6(x) \end{bmatrix} \delta \left(x - \frac{l}{2} \right) f(t) dx = \frac{f(t)}{8} \begin{bmatrix} 4 \\ l \\ \hline 4 \\ -l \end{bmatrix}, \quad \mathbf{F}_2(t) = \begin{bmatrix} 0 \\ 0 \\ \hline 0 \\ 0 \end{bmatrix} \quad (3)$$

$\mathbf{F}_1(t)$ follows from (5-44) with the shape function $N_j(x)$ given by (5-35). At the evaluation of the integral the concentrated force at $x = \frac{l}{2}$ has formally been written in terms of an equivalent distributed load per unit length by means of the indicated Dirac delta function, cf. (4-68). Alternatively, the nodal forces $\mathbf{F}_1(t)$ from the concentrated force can be indicated directly from well-known standard results from FEM theory for plane structures. Next, the global stiffness matrix, global mass matrix and global load vector with no correction for geometrical boundaries, \mathbf{K}_0 , \mathbf{M}_0 and $\mathbf{F}_0(t)$ are assembled according to the topology of the system

$$\mathbf{K}_0 = \begin{bmatrix} \mathbf{K}_{1,11} & | & \mathbf{K}_{1,12} & | & \mathbf{0} \\ \hline - & - & + & - & - \\ \mathbf{K}_{1,12}^T & | & \mathbf{K}_{1,22} + \mathbf{K}_{2,11} & | & \mathbf{K}_{2,12} \\ \hline - & - & + & - & - \\ \mathbf{0} & | & \mathbf{K}_{2,12}^T & | & \mathbf{K}_{2,22} \end{bmatrix} \quad (4)$$

$$\mathbf{M}_0 = \begin{bmatrix} \mathbf{M}_{1,11} & | & \mathbf{M}_{1,12} & | & \mathbf{0} \\ \hline - & - & + & - & - \\ \mathbf{M}_{1,12}^T & | & \mathbf{M}_{1,22} + \mathbf{M}_{2,11} & | & \mathbf{M}_{2,12} \\ \hline - & - & + & - & - \\ \mathbf{0} & | & \mathbf{M}_{2,12}^T & | & \mathbf{M}_{2,22} \end{bmatrix}, \quad \mathbf{F}_0(t) = \frac{f(t)}{8} \begin{bmatrix} 4 \\ l \\ \hline 4 \\ -l \\ \hline 0 \\ 0 \end{bmatrix} \quad (5)$$

Introducing the global boundary conditions $\theta_A = u_A = \theta_B = u_B = 0$ leads to the following equation of motion, cf. (5-29)

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}(t) \quad (6)$$

$$\mathbf{q}(t) = \begin{bmatrix} u_C(t) \\ \theta_C(t) \end{bmatrix}, \quad \mathbf{K} = \mathbf{K}_{1,22} + \mathbf{K}_{2,11} = \frac{EI}{l^3} \begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix} \quad (7)$$

$$\mathbf{M} = \mathbf{M}_{1,22} + \mathbf{M}_{2,11} = \frac{\mu l}{420} \begin{bmatrix} 312 & 0 \\ 0 & 8l^2 \end{bmatrix}, \quad \mathbf{f}(t) = \frac{1}{8} \begin{bmatrix} 4 \\ -l \end{bmatrix} f_0 \cos(\omega t) \quad (8)$$

It is seen that the global mass matrix and the stiffness matrices are both diagonal. Hence, $u_C(t)$ and $\theta_C(t)$ may be interpreted as undamped modal coordinates. The undamped circular eigenfrequencies become

$$\left. \begin{aligned} \omega_1^2 &= 24 \cdot \frac{420}{312} \frac{EI}{\mu l^4} \\ \omega_1^2 &= 8 \cdot \frac{420}{8} \frac{EI}{\mu l^4} \end{aligned} \right\} \Rightarrow$$

$$\omega_j = \begin{cases} \sqrt{\frac{420}{13}} \sqrt{\frac{EI}{\mu l^4}} = 5.683986 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 1 \\ \sqrt{420} \sqrt{\frac{EI}{\mu l^4}} = 20.493902 \sqrt{\frac{EI}{\mu l^4}} & , \quad j = 2 \end{cases} \quad (8)$$

As expected the numerically calculated circular eigenfrequencies are upper bounds to the analytical solutions as given by eq. (7) of problem 3, cf. the comments subsequent to (5-28). The eigenmodes become

$$\Phi^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , \quad \Phi^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (9)$$

The eigenmodes have been sketched below in fig. 2.

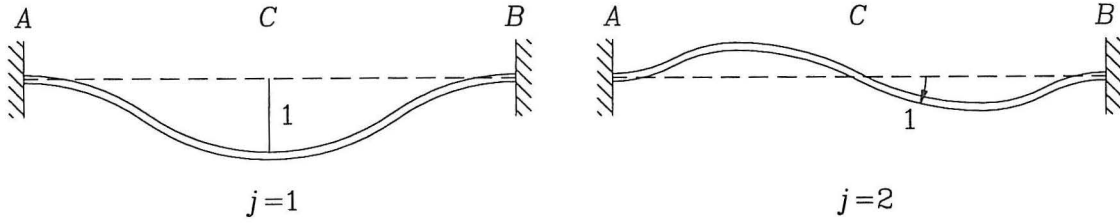


Fig. 2: Undamped eigenmodes of two-element model.

Question 2:

The system is free of damping, which means that the response and the excitation will be in phase. The stationary motion of the structure is then given as, cf. (3-100), (3-101)

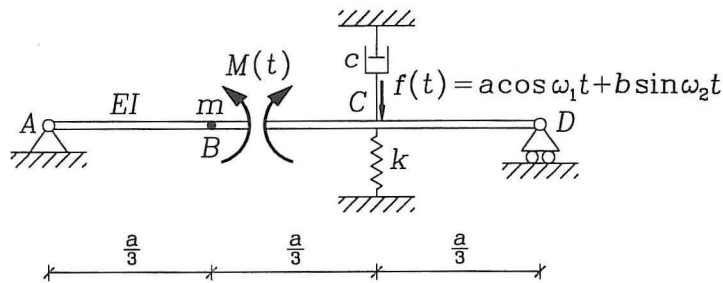
$$\mathbf{q}(t) = \mathbf{Q} \cos(\omega t) \quad (10)$$

$$\mathbf{Q} = \mathbf{H}(\omega) \frac{f_0}{8} \begin{bmatrix} 4 \\ -l \end{bmatrix} = \left[\frac{EI}{l^3} \begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix} - \frac{\mu l \omega^2}{420} \begin{bmatrix} 312 & 0 \\ 0 & 8l^2 \end{bmatrix} \right]^{-1} \frac{f_0}{8} \begin{bmatrix} 4 \\ -l \end{bmatrix} =$$

$$\frac{f_0}{8} \begin{bmatrix} \frac{1}{\frac{312}{420} \mu l (\omega_1^2 - \omega^2)} & 0 \\ 0 & \frac{1}{\frac{8}{420} \mu l^2 (\omega_2^2 - \omega^2)} \end{bmatrix} \begin{bmatrix} 4 \\ -l \end{bmatrix} = \frac{f_0}{\mu l} \begin{bmatrix} \frac{210}{312(\omega_1^2 - \omega^2)} \\ \frac{105}{16l(\omega_2^2 - \omega^2)} \end{bmatrix} \quad (11)$$

2.18 August 26, 1996

Duration: 4h

PROBLEM 1 (25%. $\mu = 8.4\%$)

The figure shows a plane linear elastic horizontal massless Bernoulli-Euler beam $ABCDE$, free of damping, with the constant bending stiffness and the total length a . The beam is simply supported at the points A and D . At the 3rd point B a point mass m is attached. At the other 3rd point C a linear viscous damping element with the damping constant c and a linear elastic spring with the spring constant k are attached, which both are acting in the vertical direction. Besides, a vertical dynamic force $f(t) = a \cos(\omega_1 t) + b \sin(\omega_2 t)$, where a and b are real constants, is acting at point C . The beam is assumed to be infinitely stiff against axial deformations, and only small vibrations from the static equilibrium state are considered.

Question 1 (5%)

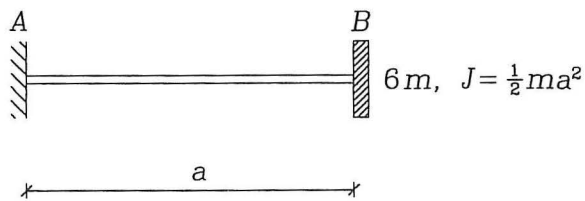
Formulate the equations of motion for the vertical displacements of the points B and C .

Question 2 (10%)

Determine the stationary motion of point B from the load $f(t)$, if it is assumed that the response from possible initial conditions has died away.

Question 3 (10%)

Determine the stationary dynamic bending moment $M(t)$ at point B from the load $f(t)$, if it is assumed that the response from possible initial conditions has died away. The sign of the bending moment has been defined in the figure.

PROBLEM 2 (25%. $\mu = 17.5\%$)

The figure shows a plane, horizontal, massless, linear elastic Bernoulli-Euler beam with the constant bending stiffness EI and the length a . The beam is fixed at point A , and at point B a distributed mass of magnitude $6m$ and with the mass moment of inertia $J = \frac{1}{2}ma^2$ is placed. The beam is assumed to be infinitely stiff against axial deformations, and only small vibrations from the static equilibrium state are considered.

Question 1 (5%)

Determine the mass matrix and the stiffness matrix of the system.

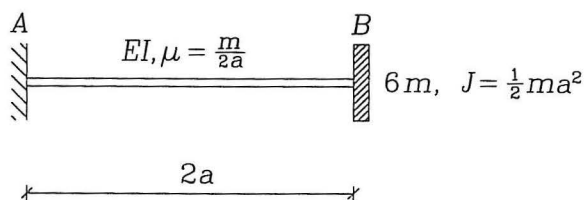
Question 2 (10%)

Determine the undamped circular eigenfrequencies and eigenmodes of the system.

Question 3 (10%)

The structure is assumed to be linearly viscous damped. Calculate the damping matrix C , when the modal damping ratios have been measured to $\zeta_1 = 1\%$ and $\zeta_2 = 2\%$, and modal decoupling can be assumed, argued by the well separated circular eigenfrequencies.

PROBLEM 3

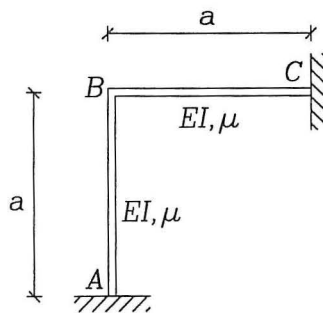


The figure shows a horizontal, plane Bernoulli-Euler of the length $2a$, free of damping and with the constant bending stiffness EI and the constant mass per unit length $\mu = \frac{m}{2a}$. The beam is fixed at the point A , and at point B a distributed mass of magnitude $6m$ and with the mass moment of inertia $J = \frac{1}{2}ma^2$ is placed. The beam is assumed to be infinitely stiff against axial deformations. Only small vertical vibrations from the static equilibrium state is considered, and the influence from a possible axial force on the response is ignored.

Question (20%. $\mu = 16.5\%$)

Formulate the frequency condition for the determination of the undamped circular eigenfrequencies of the beam. Numerical solution of the frequency condition is not required.

PROBLEM 4



The figure shows the same plane frame structure made up of the vertical beam AB and the horizontal beam BC . Both of the beam elements are linear elastic Bernoulli-Euler beams free of damping, with the constant bending stiffness EI , the constant mass per unit length μ and the length a . Beam AB is fixed at point A and beam BC is fixed at point B . Both sub-beams are assumed to be infinitely stiff against axial deformations, and only small vibrations in the plane of the structure are considered.

Question (30%. $\mu = 18.7\%$)

Formulate a finite element model with two beam elements of the structure using the two sub-beams as elements, and determine the undamped circular eigenfrequency of the structure.

SOLUTIONS

PROBLEM 1

Question 1:

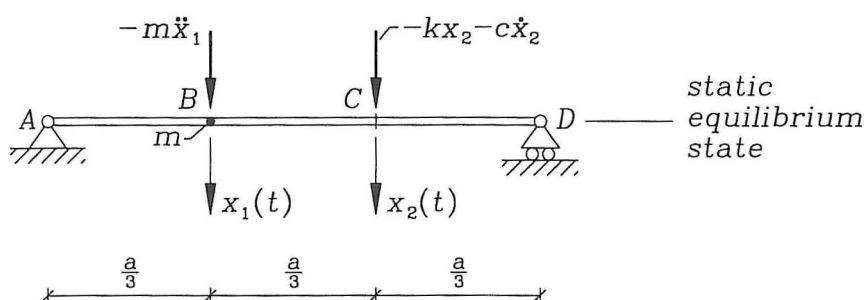


Fig. 1: Forces on a free beam.

The beam is massless. Hence, the system has but a single degree of freedom, which is selected as the vertical displacement $x_1(t)$ of point B from the static equilibrium state with the sign defined in fig. 1. Besides, an auxiliary degree of freedom \bar{x}_2 for the vertical displacement of point C from the static equilibrium state is introduced. The beam is cut free from the damper and the spring, and the damper force $c\dot{x}_2(t)$ and the spring force $kx_2(t)$ are applied as external forces at point C with the signs defined in fig. 1 along with the external force $f(t)$. Further, the inertial force $-m\ddot{x}_1(t)$ is applied as external force according to d'Alembert's principle. The equation of motion for the vertical motion of the points B and C then reads, cf. (3-1)

$$\begin{cases} x_1(t) = \delta_{11}(-m\ddot{x}_1) + \delta_{12}(f(t) - c\dot{x}_2 - kx_2) \\ x_2(t) = \delta_{21}(-m\ddot{x}_1) + \delta_{22}(f(t) - c\dot{x}_2 - kx_2) \end{cases} \quad (1)$$

The flexibility matrix is given as, cf. (B-1)

$$\mathbf{D} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \frac{1}{486} \frac{a^3}{EI} \begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix} \quad (2)$$

The equations of motion can then be written

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \quad (3)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = m \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C} = c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (4)$$

$$\mathbf{K} = \frac{162}{5} \frac{EI}{a^3} \begin{bmatrix} 8 & -7 \\ -7 & 8 + \kappa \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t) \quad (5)$$

$$\kappa = \frac{5}{162} \frac{a^3 k}{EI} \quad (6)$$

Question 2:

Since, $\cos(\omega_1 t) = \text{Re}(\exp(i\omega_1 t))$ and $\sin(\omega_2 t) = \text{Re}(-i \exp(i\omega_2 t))$, the load vector can be written

$$\begin{cases} \mathbf{f}(t) = \text{Re}(\mathbf{F}_1 e^{i\omega_1 t} + \mathbf{F}_2 e^{i\omega_2 t}) \\ \mathbf{F}_1 = a \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{F}_2 = -ib \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases} \quad (7)$$

The stationary response then becomes, cf. (3-108), (3-109)

$$\mathbf{x}(t) = \text{Re}(\mathbf{X}_1 e^{i\omega_1 t} + \mathbf{X}_2 e^{i\omega_2 t}) \quad (8)$$

$$\left. \begin{aligned} \mathbf{X}_1 &= \begin{bmatrix} X_{1,1} \\ X_{1,2} \end{bmatrix} = \mathbf{H}(\omega_1) \mathbf{F}_1 = \mathbf{H}_2(\omega_1) a \\ \mathbf{X}_2 &= \begin{bmatrix} X_{2,1} \\ X_{2,2} \end{bmatrix} = \mathbf{H}(\omega_2) \mathbf{F}_2 = -i \mathbf{H}_2(\omega_2) b \end{aligned} \right\} \quad (9)$$

where $\mathbf{H}_2(\omega)$ signifies the second column of the frequency response matrix, cf. remarks subsequent to (3-104). $\mathbf{H}_2(\omega)$ is given as, cf. (3-102)

$$\mathbf{H}_2(\omega) = \frac{5}{162} \frac{a^3}{EI} \begin{bmatrix} 8 - \lambda & -7 \\ -7 & 8 + \kappa + i\gamma \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{5}{162} \frac{a^3}{EI} \frac{1}{D} \begin{bmatrix} 7 \\ 8 - \lambda \end{bmatrix} \quad (10)$$

$$\lambda = \frac{5}{162} \frac{m\omega^2 a^3}{EI}, \quad \gamma = \frac{5}{162} \frac{c\omega a^3}{EI}, \quad D = (8 + \lambda)(8 + \kappa + i\gamma) - 49 \quad (11)$$

Question 3:

The bending moment at point B is given as

$$M(t) = \delta_{M1}(-m\ddot{x}_1) + \delta_{M2}(f(t) - c\dot{x}_2 - kx_2) \quad (12)$$

The coefficients of influence δ_{M1} and δ_{M2} for the bending moment from static unit forces at x_1 and x_2 are given as

$$\delta_{M1} = \frac{2}{9}l, \quad \delta_{M2} = \frac{1}{9}l \quad (13)$$

The stationary bending moment is made up of a sum of harmonic contributions with the circular frequencies ω_1 and ω_2 similar to the the displacement response (8), i.e.

$$M(t) = \text{Re}(M_1 e^{i\omega_1 t} + M_2 e^{i\omega_2 t}) \quad (14)$$

Insertion of (7), (8), (9), (13) into (12) provides the following solution for the complex amplitudes M_1 and M_2 of the harmonic components of $M(t)$

$$\left. \begin{aligned} M_1 &= \frac{2}{9}l(m\omega^2 X_{1,1}) + \frac{1}{9}l(a - (ic\omega + k)X_{1,2}) \\ M_2 &= \frac{2}{9}l(m\omega^2 X_{2,1}) + \frac{1}{9}l(-ib - (ic\omega + k)X_{2,2}) \end{aligned} \right\} \quad (15)$$

PROBLEM 2

Question 1:

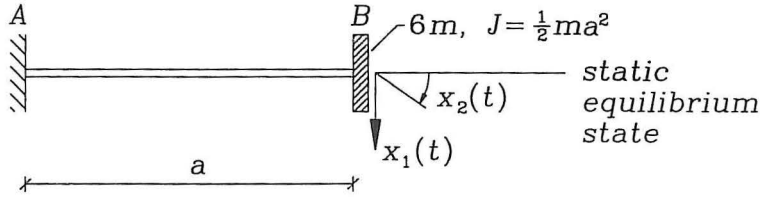


Fig. 1: Definition of degrees of freedom.

The beam is massless. Hence, the system has two degrees of freedom, which are selected as the vertical displacement $x_1(t)$ and the rotation $x_2(t)$ of point B in clock-wise direction from the static equilibrium state with the signs defined in fig. 1. The equations of motion for undamped eigen vibrations can then be written, cf. (5-61)

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \quad (1)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = \frac{1}{2}m \begin{bmatrix} 12 & 0 \\ 0 & a^2 \end{bmatrix}, \quad \mathbf{K} = \frac{EI}{a^3} \begin{bmatrix} 12 & -6a \\ -6a & 4a^2 \end{bmatrix} \quad (2)$$

Question 2:

The undamped circular eigenfrequencies and eigenmodes are obtained as non-trivial solutions to the linear eigenvalue problem, cf. (3-42)

$$\begin{bmatrix} 6 - 12\lambda_j & -3a \\ -3a & (2 - \lambda_j)a^2 \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3)$$

$$\lambda_j = \frac{1}{4} \frac{ma^3}{EI} \omega_j^2 \quad (4)$$

The characteristic equation becomes

$$(6 - 12\lambda_j)((2 - \lambda_j)a^2) - 9a^2 = 0 \Rightarrow$$

$$\lambda_j = \begin{cases} \frac{1}{4}(5 - \sqrt{21}) & , \quad j = 1 \\ \frac{1}{4}(5 + \sqrt{21}) & , \quad j = 2 \end{cases} \quad (5)$$

$$\omega_j = \begin{cases} \sqrt{5 - \sqrt{21}} \sqrt{\frac{EI}{ma^3}} & , \quad j = 1 \\ \sqrt{5 + \sqrt{21}} \sqrt{\frac{EI}{ma^3}} & , \quad j = 2 \end{cases} \quad (6)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix} \quad (8)$$

The displacement component $\Phi_1^{(j)}$ is determined from the 2nd equation in (3)

$$\Phi_1^{(j)} = \frac{1}{3}(2 - \lambda_j)a = \begin{cases} \frac{1}{12}(3 + \sqrt{21})a & , \quad j = 1 \\ \frac{1}{12}(3 - \sqrt{21})a & , \quad j = 2 \end{cases} \quad (9)$$

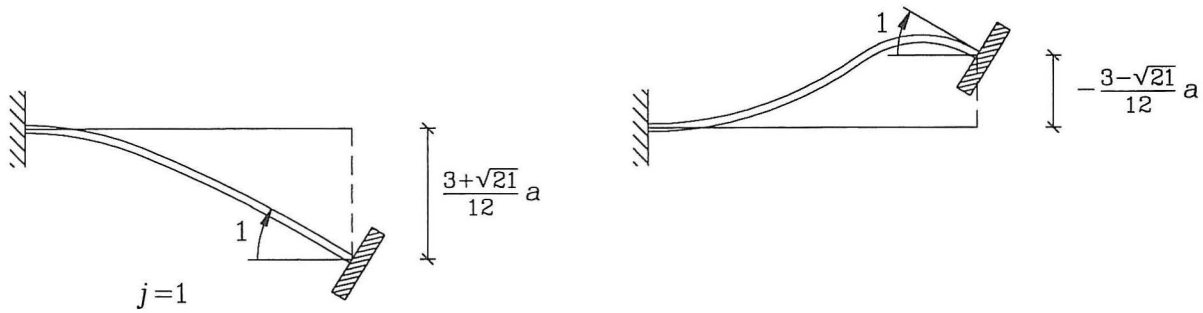


Fig. 2: Eigenmodes of two degrees-of-freedom system.

Question 3:

The damping matrix can be represented by the following Rayleigh damping model, cf. (3-284), (2)

$$\mathbf{C} = a_0 \mathbf{M} + a_1 \mathbf{K} = a_0 \frac{m}{2} \begin{bmatrix} 12 & 0 \\ 0 & a^2 \end{bmatrix} + a_1 \frac{EI}{a^3} \begin{bmatrix} 12 & -6a \\ -6a & 4a^2 \end{bmatrix} \quad (10)$$

where the expansion coefficients a_0 and a_1 are calibrated from the linear equations, cf. (3-287), (6)

$$\begin{aligned} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} &= \frac{2\omega_1\omega_2}{\omega_2^2 - \omega_1^2} \begin{bmatrix} \omega_2 & -\omega_1 \\ -\frac{1}{\omega_2} & \frac{1}{\omega_1} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = 2\sqrt{1 + \frac{5}{\sqrt{21}}} \begin{bmatrix} \left(\zeta_1 - \frac{5-\sqrt{21}}{2}\zeta_2\right) \sqrt{\frac{EI}{ma^3}} \\ \left(\zeta_2 - \frac{5-\sqrt{21}}{2}\zeta_1\right) \sqrt{\frac{ma^3}{4EI}} \end{bmatrix} \Rightarrow \\ \mathbf{C} &= \sqrt{1 + \frac{5}{\sqrt{21}}} \left(\left(\zeta_1 - \frac{5-\sqrt{21}}{2}\zeta_2\right) \begin{bmatrix} 12 & 0 \\ 0 & a^2 \end{bmatrix} + \left(\zeta_2 - \frac{5-\sqrt{21}}{2}\zeta_1\right) \begin{bmatrix} 12 & -6a \\ -6a & 4a^2 \end{bmatrix} \right) \sqrt{\frac{mEI}{a^3}} \quad (11) \end{aligned}$$

PROBLEM 3

The beam has constant bending stiffness EI , constant mass per unit length $\mu = \frac{m}{2a}$ and the axial force in the static equilibrium state $N = 0$. Hence, the eigenmode is given by (4-18), (4-19)

$$\Phi(x) = A \sin\left(\lambda \frac{x}{2a}\right) + B \cos\left(\lambda \frac{x}{2a}\right) + C \sinh\left(\lambda \frac{x}{2a}\right) + D \cosh\left(\lambda \frac{x}{2a}\right) \quad (1)$$

$$\lambda^4 = \frac{\mu \omega^2 (2a)^4}{EI} = 8 \frac{m \omega^2 a^3}{EI} \quad (2)$$

where x is a coordinate along the beam measured from A . The boundary conditions become, cf. (4-13)

$$\left. \begin{aligned} \Phi(0) = \frac{d}{dx}\Phi(0) &= 0 \\ \frac{d^2}{dx^2}\Phi(2a) &= \omega^2 \frac{J}{EI} \frac{d}{dx}\Phi(2a) = \frac{1}{16a} 8 \frac{m \omega^2 a^3}{EI} \frac{d}{dx}\Phi(2a) = \frac{1}{16a} \lambda^4 \frac{d}{dx}\Phi(2a) \\ \frac{d^3}{dx^3}\Phi(2a) &= -\omega^2 \frac{6m}{EI} \Phi(2a) = -\frac{3}{4a^3} 8 \frac{m \omega^2 a^3}{EI} \Phi(2a) = -\frac{3}{4a^3} \lambda^4 \Phi(2a) \end{aligned} \right\} \quad (3)$$

where ω^2 has been eliminated in favour of λ^4 by means of (2). The boundary conditions at $x = 0$ provide $D = -B$ and $C = -A$, cf. (4-37), leading to the following reduced expression for the eigenmode

$$\Phi(x) = A \left(\sin\left(\lambda \frac{x}{2a}\right) - \sinh\left(\lambda \frac{x}{2a}\right) \right) + B \left(\cos\left(\lambda \frac{x}{2a}\right) - \cosh\left(\lambda \frac{x}{2a}\right) \right) \quad (4)$$

Next, (4) is inserted into the boundary conditions at point B at $x = 2a$, leading to the following homogeneous linear equations

$$\begin{bmatrix} -\frac{\lambda^2}{4}(\sin \lambda + \sinh \lambda) - \frac{\lambda^5}{32}(\cos \lambda - \cosh \lambda) & -\frac{\lambda^2}{4}(\cos \lambda + \cosh \lambda) + \frac{\lambda^5}{32}(\sin \lambda + \sinh \lambda) \\ -\frac{\lambda^3}{8}(\cos \lambda + \cosh \lambda) + \frac{3\lambda^4}{4}(\sin \lambda - \sinh \lambda) & \frac{\lambda^3}{8}(\sin \lambda - \sinh \lambda) + \frac{3\lambda^4}{4}(\cos \lambda - \cosh \lambda) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (5)$$

The frequency condition is obtained from setting the determinant of the coefficient matrix equal to zero. Although it will not be attempted here, the resulting equation can be reduced significantly by use of trigonometric and hyperbolic identities. The lowest 4 solutions of the frequency conditions become

$$\lambda_j = \begin{cases} 0.823614 & , \quad j = 1 \\ 2.347782 & , \quad j = 2 \\ 4.837720 & , \quad j = 3 \\ 7.890459 & , \quad j = 4 \end{cases} \Rightarrow \quad (6)$$

$$\omega_j = \begin{cases} 0.23933 \sqrt{\frac{EI}{ma^3}} & , \quad j = 1 \\ 1.94882 \sqrt{\frac{EI}{ma^3}} & , \quad j = 2 \\ 8.27440 \sqrt{\frac{EI}{ma^3}} & , \quad j = 3 \\ 22.01201 \sqrt{\frac{EI}{ma^3}} & , \quad j = 4 \end{cases} \quad (7)$$

Use of consistent mass matrix for the continuous part of the beam AB with $\mu = \frac{m}{2a}$ and $l = 2a$ results in the following mass global mass and stiffness matrices with the degrees of freedom defined in fig. 5.5 in the textbook, cf. (5-60), (5-61)

$$\left. \begin{aligned} \mathbf{M} &= \frac{m}{420} \begin{bmatrix} 156 & -44a \\ -44a & 16a^2 \end{bmatrix} + \frac{m}{2} \begin{bmatrix} 12 & 0 \\ 0 & a^2 \end{bmatrix} = \frac{m}{420} \begin{bmatrix} 2676 & -44a \\ -44a & 226a^2 \end{bmatrix} \\ \mathbf{K} &= \frac{EI}{8a^3} \begin{bmatrix} 12 & -12a \\ -12a & 16a^2 \end{bmatrix} \end{aligned} \right\} \quad (8)$$

The circular eigenfrequencies of the indicated discrete system become

$$\omega_j = \begin{cases} 0.23983 \sqrt{\frac{EI}{ma^3}} & , \quad j = 1 \\ 1.95332 \sqrt{\frac{EI}{ma^3}} & , \quad j = 2 \end{cases} \quad (9)$$

As expected (9) represents upper bounds to the corresponding analytical results. However, the approximation is pretty good in this because the distributed mass of the beam element is relatively small compared to the discrete contribution at the end of the beam.

PROBLEM 4

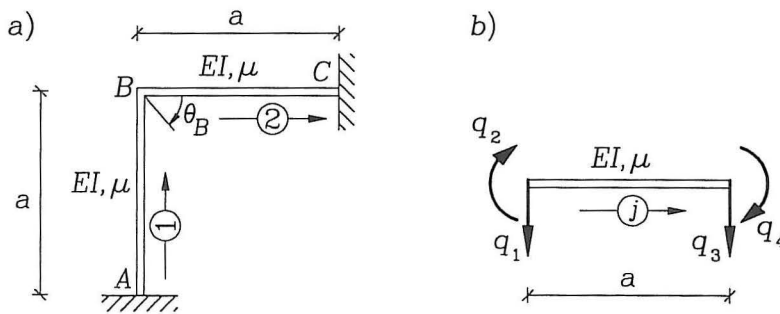


Fig. 1: Finite element model with two beam elements. a) Definition of global degree of freedom and structural data. b) Definition of element local degrees of freedom of element j .

The continuous system is modelled by two finite elements identical to the beam elements AB and BC , respectively. Because of the geometrical boundary conditions at the points A and C and because the beams have been assumed to be infinitely stiff against axial deformations, the discrete system has but a single global degree of freedom, which is selected as the rotation θ_B of point B . The local degrees of freedom of the two elements

with identical mechanical properties have been indicated in fig. 1b. The corresponding element stiffness matrices and consistent mass matrices in local coordinates become, cf. (5-42), (5-43)

$$\mathbf{k}_j = \frac{EI}{a^3} \begin{bmatrix} 12 & 6a & -12 & 6a \\ 6a & 4a^2 & -6a & 2a^2 \\ -12 & -6a & 12 & -6a \\ 6a & 2a^2 & -6a & 4a^2 \end{bmatrix}, \quad j = 1, 2 \quad (1)$$

$$\mathbf{m}_j = \frac{\mu a}{420} \begin{bmatrix} 156 & 22a & 54 & -13a \\ 22a & 4a^2 & 13a & -3a^2 \\ 54 & 13a & 156 & -22a \\ -13a & -3a^2 & -22a & 4a^2 \end{bmatrix}, \quad j = 1, 2 \quad (2)$$

Because of the geometrical constraints only the elements $k_{1,44}$ and $k_{2,22}$ of the local stiffness matrices for the elements 1 and 2 will affect the elastic restoring of node B . Similarly, only the elements $m_{1,44}$ and $m_{2,22}$ of the local mass matrices will affect the inertial moment loading of node B . The global equation of motion then becomes

$$m\ddot{\theta}_B + k\theta_B = 0 \quad (3)$$

$$k = k_{1,44} + k_{2,22} = 4a^2 \frac{EI}{a^3} + 4a^2 \frac{EI}{a^3} = 8 \frac{EI}{a} \quad (4)$$

$$m = m_{1,44} + m_{2,22} = 4a^2 \frac{\mu a}{420} + 4a^2 \frac{\mu a}{420} = \frac{2}{105} \mu a^3 \quad (5)$$

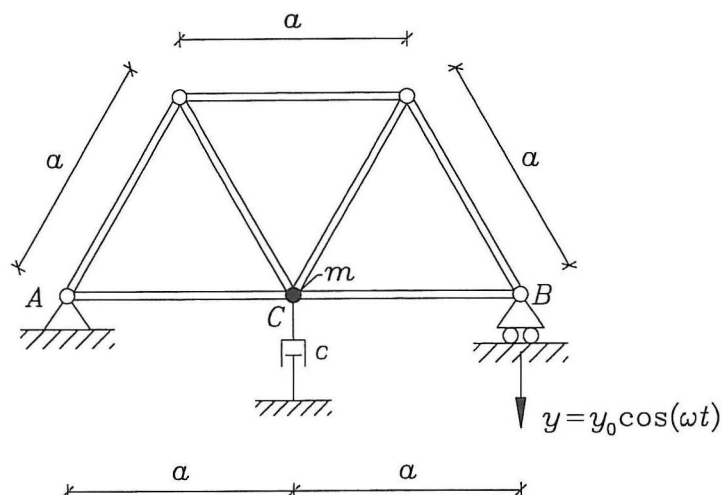
The circular eigenfrequency then follows from (2-7)

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{420} \sqrt{\frac{EI}{\mu a^4}} \quad (6)$$

2.19 June 19, 1997

Duration: 4h

PROBLEM 1



The figure shows a plane truss, where the nodes A , B and C are placed on the same horizontal line. All bars have the length a and are assumed to be linearly elastic with the constant axial stiffness AE , where A is the cross-sectional area and E is the elasticity modulus. Besides, the bars are assumed to be massless and free of damping. The structure is simply supported at the points A and B . At point C a point mass m and a linearly viscous damper with the damping constant c are attached. The damping element is only active against vertical motions of point C .

The structure is excited by a vertical harmonically varying motion $y(t) = y_0 \cos(\omega t)$ with the amplitude y_0 and the circular frequency ω of the support at point B . The support at point A and the support of the viscous damping element at point C are both assumed to be fixed. Only small motions from the static equilibrium state in the plane of the structure are considered.

Question 1 (20%. $\mu = 9.8\%$)

Formulate the equations of motion for the mass at point C .

Question 2 (15%. $\mu = 7.3\%$)

Determine the undamped circular eigenfrequencies and eigenmodes, and the modal damping ratios of the structure.

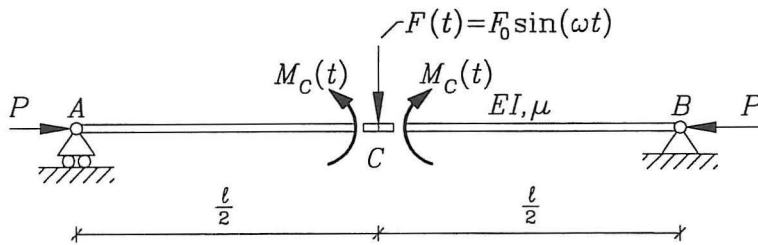
Question 3 (10%. $\mu = 4.9\%$)

Determine the motion of point C from the static equilibrium state, after the excitation has been acting for infinitely long so the response from possible initial values has died away.

Help:

A vertical force V in the downward direction at point C causes a vertical displacement of point C in the same direction of magnitude $\frac{11}{6} \frac{aV}{AE}$, and a horizontal displacement of point C to the right of the magnitude $\frac{\sqrt{3}}{6} \frac{aV}{AE}$. A horizontal force H at point C causes a horizontal displacement of point C in the same direction of magnitude $\frac{aH}{AE}$.

PROBLEM 2

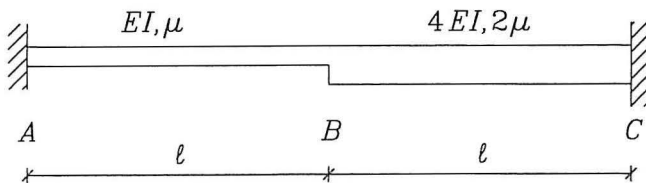


The figure shows a plane horizontal, rectilinear Bernoulli-Euler beam ACB of the length l , with the constant bending stiffness EI , with the constant mass per unit length μ and free from damping. The beam is simply supported at the points A and B . In the static equilibrium state the beam is loaded with a compressive axial force P as shown in the figure. Additionally, the beam is loaded at the midpoint C by a vertical harmonically varying concentrated force $F(t) = F_0 \sin(\omega t)$ with the amplitude F_0 and the circular frequency ω . The beam is assumed to be infinitely stiff against axial deformations, and only small vertical vibrations from the static equilibrium state are considered.

Question (25%. $\mu = 12.7\%$)

Determine the time-variation of the bending moment at point C in the stationary state, where the response from possible initial conditions has died away.

PROBLEM 3



The figure shows a plane, horizontal, rectilinear beam structure made up of the Bernoulli-Euler beams AB and BC . Both sub-beams have the length l , and are free of damping. Beam AB has the constant bending stiffness EI and the constant mass per unit length μ , and beam BC has the constant bending stiffness $4EI$ and the constant mass per unit length 2μ . The structure is fixed at the points A and C . The beams are infinitely stiff against axial deformations, and only small vertical vibrations from the static equilibrium state are considered. Additionally, the influence from possible axial forces on the dynamic response is ignored.

Question 1 (20%. $\mu = 10.1\%$)

Formulate a finite element model with two beam elements of the structure using two sub-beams as elements by calculating the global stiffness matrix and global consistent mass matrix with due consideration to the geometric boundary conditions.

Question 2 (10%. $\mu = 2.9\%$)

Use the formulated finite element model to estimate the two lowest undamped circular eigenfrequencies of the structure.

SOLUTIONS

PROBLEM 1

Question 1:

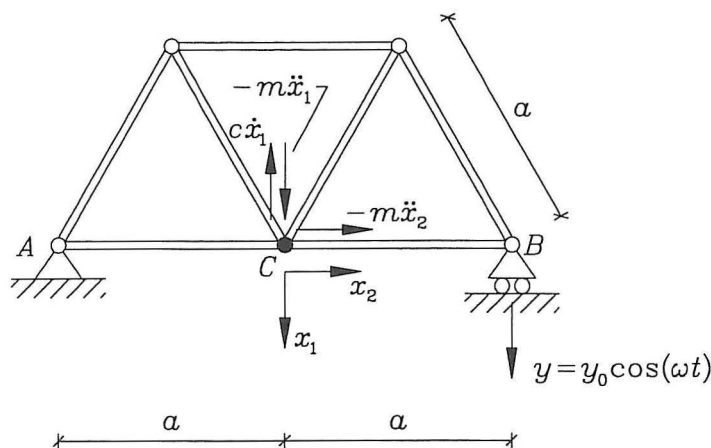


Fig. 1: Definition of degrees of freedom. Forces on mass.

Since all bars are assumed to be massless and free of damping, the system has but 2 dynamic degrees of freedom which are selected as the vertical displacement $x_1(t)$ and the horizontal displacements $x_2(t)$ of point C from the statical equilibrium state. These are made up of quasi-static components $x_1^{(0)}(t)$ and $x_2^{(0)}(t)$ caused by the stiff-body motion of the structure from the motion of the support at point B and dynamic components from the inertial loads and the damping load at point C . The quasi-static motion can be written

$$\begin{bmatrix} x_1^{(0)}(t) \\ x_2^{(0)}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} y(t) \quad (1)$$

The structure is cut free from the damper, and the damper force $c\dot{x}_1(t)$ is applied as an external force with a sign as defined in fig. 1. Further, the inertial loads $-m\ddot{x}_1(t)$ and $-m\ddot{x}_2(t)$ are applied in accordance with d'Alembert's principle. The equations of motion then read, cf. (3-328)

$$\begin{aligned} x_1(t) &= x_1^{(0)}(t) + \delta_{11}(-m\ddot{x}_1 - c\dot{x}_1) + \delta_{12}(-m\ddot{x}_2) \\ x_2(t) &= x_2^{(0)}(t) + \delta_{21}(-m\ddot{x}_1 - c\dot{x}_1) + \delta_{22}(-m\ddot{x}_2) \end{aligned} \quad (2)$$

The flexibility coefficients are given immediately after the problem test. The flexibility matrix becomes

$$\mathbf{D} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \frac{1}{6} \frac{a}{AE} \begin{bmatrix} 11 & \sqrt{3} \\ \sqrt{3} & 6 \end{bmatrix} \quad (3)$$

The equations of motion for the displacement of point C can then be written in the standard form as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}_0 \cos \omega t \quad (4)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \quad (5)$$

$$\mathbf{K} = \mathbf{D}^{-1} = \frac{AE}{a} \frac{2}{21} \begin{bmatrix} 6 & -\sqrt{3} \\ -\sqrt{3} & 11 \end{bmatrix}, \quad \mathbf{f}_0 = \mathbf{K} \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} y_0 = \frac{1}{21} \frac{AE}{a} \begin{bmatrix} 6 \\ -\sqrt{3} \end{bmatrix} y_0 \quad (6)$$

Question 2:

Undamped circular eigenfrequencies and eigenmodes $\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix}$ become, cf. (3-42)

$$\begin{bmatrix} 6 - \lambda_j & -\sqrt{3} \\ -\sqrt{3} & 11 - \lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (7)$$

$$\lambda_j = \frac{21}{2} \frac{ma}{AE} \omega_j^2 \quad (8)$$

The characteristic equation then becomes

$$(6 - \lambda_j)(11 - \lambda_j) - 3 = 0 \Rightarrow$$

$$\lambda_j = \begin{cases} \frac{1}{2}(17 - \sqrt{37}) & , \quad j = 1 \\ \frac{1}{2}(17 + \sqrt{37}) & , \quad j = 2 \end{cases} \quad (9)$$

$$\omega_j = \begin{cases} \sqrt{\frac{17}{21} - \frac{\sqrt{37}}{21}} \sqrt{\frac{AE}{ma}} & , \quad j = 1 \\ \sqrt{\frac{17}{21} + \frac{\sqrt{37}}{21}} \sqrt{\frac{AE}{ma}} & , \quad j = 2 \end{cases} \quad (10)$$

The eigenmodes are normalized as follows

$$\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix} \quad (11)$$

The first component $\Phi_1^{(j)}$ is determined from the first equation of (7) as follows

$$(6 - \lambda_j)\Phi_1^{(j)} - \sqrt{3} \cdot 1 = 0 \Rightarrow$$

$$\Phi_1^{(j)} = \frac{\sqrt{3}}{6 - \lambda_j} = \begin{cases} \frac{\sqrt{3}}{6}(5 + \sqrt{37}) & , \quad j = 1 \\ \frac{\sqrt{3}}{6}(5 - \sqrt{37}) & , \quad j = 2 \end{cases} \quad (12)$$

The modal damping ratios become, cf. (3-182), (3-183)

$$\begin{aligned} \zeta_j &= \frac{\Phi^{(j)T} \mathbf{C} \Phi^{(j)}}{2\omega_j \Phi^{(j)T} \mathbf{M} \Phi^{(j)}} = \frac{(\Phi_1^{(j)})^2 c}{2\omega_j m ((\Phi_1^{(j)})^2 + 1)} = \\ &= \frac{\sqrt{\frac{189}{8}}}{\sqrt{\lambda_j}(\lambda_j^2 - 12\lambda_j + 39)} \sqrt{\frac{ac^2}{mAE}} = \begin{cases} 0.6317430 \sqrt{\frac{ac^2}{mAE}} & , \quad j = 1 \\ 0.0424461 \sqrt{\frac{ac^2}{mAE}} & , \quad j = 2 \end{cases} \end{aligned} \quad (13)$$

where (8), (9) and (12) have been used.

Question 3:

The stationary response of (4) becomes, see (3-100), (3-101)

$$\mathbf{x}(t) = \text{Re}(\mathbf{X}e^{i\omega t}) \quad (14)$$

$$\mathbf{X} = \mathbf{H}(\omega)\mathbf{f}_0 \quad (15)$$

where \mathbf{f}_0 is given by (16) and the frequency response matrix $\mathbf{H}(\omega)$ is given by, cf. (3-102),

$$\mathbf{H}(\omega) = (\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{C})^{-1} = \begin{bmatrix} \frac{12}{21} \frac{AE}{a} - m\omega^2 + i\omega c & -\frac{2\sqrt{3}}{21} \frac{AE}{a} \\ -\frac{2\sqrt{3}}{21} \frac{AE}{a} & \frac{22}{21} \frac{AE}{a} - m\omega^2 \end{bmatrix}^{-1} =$$

$$\frac{1}{D} \begin{bmatrix} \frac{22}{21} \frac{AE}{a} - m\omega^2 & \frac{2\sqrt{3}}{21} \frac{AE}{a} \\ \frac{2\sqrt{3}}{21} \frac{AE}{a} & \frac{12}{21} \frac{AE}{a} - m\omega^2 + i\omega c \end{bmatrix} \quad (16)$$

$$D = \left(\frac{12}{21} \frac{AE}{a} - m\omega^2 + i\omega c \right) \left(\frac{22}{21} \frac{AE}{a} - m\omega^2 \right) - \left(\frac{2\sqrt{3}}{21} \frac{AE}{a} \right)^2 \quad (17)$$

The amplitude vector (15) is written in the form

$$\mathbf{X} = \begin{bmatrix} |X_1|e^{-i\psi_1} \\ |X_2|e^{-i\psi_2} \end{bmatrix} \quad (18)$$

The displacement response (14) can then be written

$$\mathbf{x}(t) = \begin{bmatrix} \text{Re}(|X_1|e^{-i\psi_1}e^{i\omega t}) \\ \text{Re}(|X_2|e^{-i\psi_2}e^{i\omega t}) \end{bmatrix} = \begin{bmatrix} |X_1|\cos(\omega t - \psi_1) \\ |X_2|\cos(\omega t - \psi_2) \end{bmatrix} \quad (19)$$

The amplitudes $|X_i|$ and phases ψ_i are determined, equating (15) and (18). In doing this it should be noticed that the determinant D is complex.

PROBLEM 2

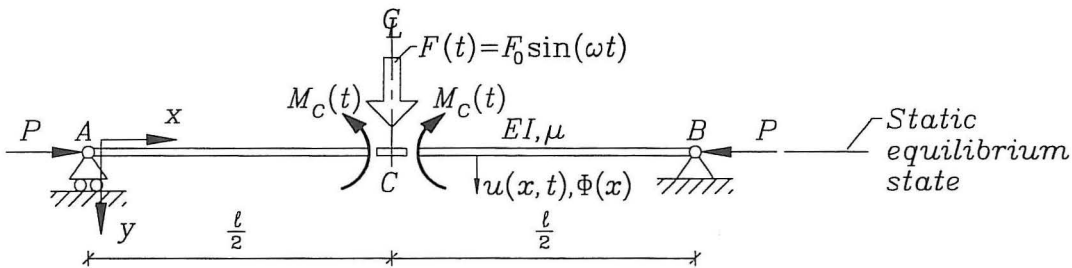


Fig. 1: Definition of system.

The undamped circular eigenfrequencies become, see (4-32), (4-33), (4-34)

$$\omega_j = \omega_{j,0} \sqrt{1 - \frac{1}{j^2} \frac{P}{P_E}} \quad , \quad \omega_{j,0} = j^2 \pi^2 \sqrt{\frac{EI}{\mu l^4}} \quad , \quad P_E = \pi^2 \frac{EI}{l^2} \quad (1)$$

The eigenmodes become, see (4-31)

$$\Phi^{(j)}(x) = \sin \left(j \pi \frac{x}{l} \right) \quad (2)$$

With the normalization of the eigenmodes as follows from (2), the modal masses become, cf. (4-64), (4-65)

$$M_j = \int_0^l \mu \left(\Phi^{(j)}(x) \right)^2 dx = \frac{1}{2} \mu l \quad , \quad j = 1, 2, \dots \quad (3)$$

The vertical displacement $u(x, t)$ co-directional to the y -axis is given by the following modal expansion, cf. (4-52)

$$u(x, t) = \sum_{j=1}^{\infty} \Phi^{(j)}(x) q_j(t) \quad (4)$$

Because the system is free of damping, the modal coordinates are obtained as solutions to the decoupled differential equations, see (4-53)

$$\ddot{q}_j + \omega_j^2 q_j = \frac{1}{M_j} F_j(t) \quad , \quad j = 1, 2, \dots \quad (5)$$

The dynamic load can formally be written as the equivalent load per unit length, cf. (4-68)

$$f_d(x, t) = \delta \left(x - \frac{l}{2} \right) F(t) \quad (6)$$

The modal loads then follow from (4-54) and (2)

$$F_j(t) = \int_0^l \Phi^{(j)}(x) f_d(x, t) dx = \Phi^{(j)} \left(\frac{l}{2} \right) F(t) = \sin \left(\frac{j\pi}{2} \right) F_0 \sin \omega t \quad , \quad j = 1, 2, \dots \quad (7)$$

With $F_j(t)$ given by (7) and use of (3) the stationary solution of (5) becomes

$$q_j(t) = Q_j(\omega) \sin(\omega t) \quad (8)$$

$$Q_j(\omega) = \frac{2 \sin\left(\frac{j\pi}{2}\right)}{\mu l \omega_j^2 - \omega^2} F_0 \quad , \quad j = 1, 2, \dots \quad (9)$$

Inserting (8) into (4) the stationary dynamic displacement field can next be determined. With the sign as defined in fig. 1, the stationary dynamic bending moment at the midpoint finally becomes, see (4-4)

$$M_C(t) = -EI \frac{\partial^2}{\partial x^2} u\left(\frac{l}{2}, t\right) = M_{C,0}(\omega) \sin(\omega t) \quad (10)$$

$$M_{C,0}(\omega) = -EI \sum_{j=1}^{\infty} \frac{d^2}{dx^2} \Phi^{(j)}\left(\frac{l}{2}\right) Q_j(\omega) = EI \sum_{j=1}^{\infty} \left(\frac{j\pi}{l}\right)^2 \sin\left(\frac{j\pi}{2}\right) \frac{2 \sin\left(\frac{j\pi}{2}\right)}{\mu l \omega_j^2 - \omega^2} F_0 =$$

$$\pi^4 \frac{EI}{\mu l^4} \sum_{j=1}^{\infty} \frac{j^2}{\pi^2} \frac{(1 - (-1)^j)}{j^4 \omega_{1,0}^2 \left(1 - \frac{1}{j^2} \frac{P}{P_E}\right) - \omega^2} F_0 l = \frac{1}{4} F_0 l D\left(\frac{\omega}{\omega_{1,0}}, \frac{P}{P_E}\right) \quad (11)$$

$$D\left(\frac{\omega}{\omega_{1,0}}, \frac{P}{P_E}\right) = \frac{4 \cdot 2}{\pi^2} \sum_{j=1,3,5,\dots} \frac{j^2}{j^4 - j^2 \frac{P}{P_E} - \frac{\omega^2}{\omega_{1,0}^2}} =$$

$$\frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(2n-1)^2}{(2n-1)^4 - (2n-1)^2 \frac{P}{P_E} - \frac{\omega^2}{\omega_{1,0}^2}} \quad (12)$$

where $\sin^2\left(\frac{j\pi}{2}\right) = \frac{1}{2}(1 - (-1)^j)$ has been inserted and $\omega_{1,0} = \pi^2 \sqrt{\frac{EI}{\mu l^4}}$ has been used to normalize the circular excitation frequency ω . $\frac{1}{4} F_0 l$ is the quasi-static moment at point C in the special case of $\frac{P}{P_E} = 0$. Hence, $D\left(\frac{\omega}{\omega_{1,0}}, 0\right)$ is a dynamic amplification factor for a case of no axial force, and one would expect $D(0, 0) = 1$. Actually, this follows from (12) by application of the series

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \dots = \frac{\pi^2}{8} \quad (13)$$

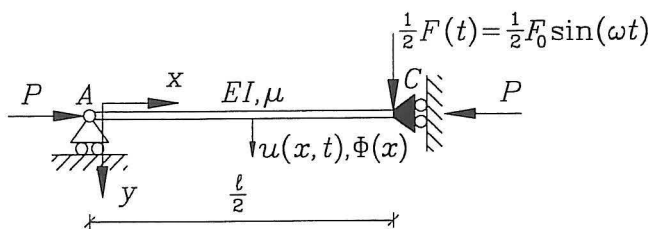


Fig. 2: Equivalent system for analysis of symmetric vibrations.

Alternatively, the stationary displacement field and the stationary bending moment field can be obtained by direct integration of the partial differential equation for the beam element, without any resort to series solutions, which follows from the modal expansion technique. The geometrical and physical symmetry of the structure around point C is utilized. In the centre line the slope and the shear force is zero. Hence, the beam can be analysed by the equivalent system shown in fig. 2. Half of the force is attached to equivalent system as sketched in fig. 1. The partial differential equation and associated boundary conditions with $N = -P$ become, cf. (4-11)

$$\left. \begin{aligned} EI \frac{\partial^4 u}{\partial x^4} + P \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial t^2} &= 0 \quad , \quad x \in \left] 0, \frac{l}{2} \right[\\ u(0, t) &= \frac{\partial^2 u(0, t)}{\partial x^2} = \frac{\partial u(\frac{l}{2}, t)}{\partial x} = 0 \\ -EI \frac{\partial^3 u(\frac{l}{2}, t)}{\partial x^3} - P \frac{\partial u(\frac{l}{2}, t)}{\partial x} &= \frac{1}{2} F_0 \sin(\omega t) \end{aligned} \right\} \quad (14)$$

Using $\frac{\partial u(\frac{l}{2}, t)}{\partial x} = 0$ the boundary condition for the shear force reduces to

$$\frac{\partial^3 u(\frac{l}{2}, t)}{\partial x^3} = -\frac{F_0}{2EI} \sin(\omega t) \quad (15)$$

The stationary displacement field must be harmonic because the system is linear. Since the system is free of damping, all mass particles must be in phase and in phase with the excitation. Hence, the stationary displacement field is given on the form

$$u(x, t) = U(x) \sin(\omega t) \quad (16)$$

Inserting (16) into (14) and (15) the real amplitude function $U(x)$ is seen to fulfil the boundary problem

$$\left. \begin{aligned} EI \frac{d^4 U(x)}{dx^4} + P \frac{d^2 U(x)}{dx^2} - \mu \omega^2 U(x) &= 0 \quad , \quad x \in \left] 0, \frac{l}{2} \right[\\ U(0) &= \frac{d^2 U(0)}{dx^2} = \frac{d}{dx} U\left(\frac{l}{2}\right) = 0 \\ \frac{d^3}{dx^3} U\left(\frac{l}{2}\right) &= -\frac{F_0}{2EI} \end{aligned} \right\} \quad (17)$$

The complementary solution of (17) can be written, cf. (4-18) and (4-19)

$$U(x) = A \sin\left(\lambda \frac{x}{l}\right) + B \cos\left(\frac{x}{l}\right) + C \sinh\left(\nu \frac{x}{l}\right) + D \cosh\left(\nu \frac{x}{l}\right) \quad (18)$$

where A , B , C , D are integration constants to be determined, and λ^2 and ν^2 are the positive roots of the quadratic equations

$$\left. \begin{aligned} \lambda^4 - \frac{Pl^2}{EI} \lambda^2 - \frac{\mu\omega^2 l^4}{EI} &= 0 \\ \nu^4 + \frac{Pl^2}{EI} \nu^2 - \frac{\mu\omega^2 l^4}{EI} &= 0 \end{aligned} \right\} \quad (19)$$

Inserting (18) into the boundary conditions of (17) at $x = 0$ provides

$$\left. \begin{aligned} B + D &= 0 \\ -\frac{\lambda^2}{l^2} B + \frac{\nu^2}{l^2} D &= 0 \end{aligned} \right\} \quad (20)$$

Since $\lambda^2 + \nu^2 \neq 0$, (20) implies $B = D = 0$. Hence, (18) reduces to

$$U(x) = A \sin\left(\lambda \frac{x}{l}\right) + C \sinh\left(\nu \frac{x}{l}\right) \quad (21)$$

Insertion of (21) into the boundary conditions of (17) at $x = \frac{l}{2}$ provides

$$\left. \begin{aligned} A \frac{\lambda}{l} \cos \frac{\lambda}{2} + C \frac{\nu}{l} \cosh \frac{\nu}{2} &= 0 \\ -A \left(\frac{\lambda}{l}\right)^3 \cos \frac{\lambda}{2} + C \left(\frac{\nu}{l}\right)^3 \cosh \frac{\nu}{2} &= -\frac{F_0}{2EI} \end{aligned} \right\} \Rightarrow$$

$$\left. \begin{aligned} A &= \frac{F_0 l^3}{2EI(\nu^2 + \lambda^2)} \frac{1}{\lambda \cos \frac{\lambda}{2}} \\ C &= -\frac{F_0 l^3}{2EI(\nu^2 + \lambda^2)} \frac{1}{\nu \cosh \frac{\nu}{2}} \end{aligned} \right\} \Rightarrow$$

$$U(x) = \frac{F_0 l^3}{2EI(\nu^2 + \lambda^2)} \left(\frac{\sin\left(\lambda \frac{x}{l}\right)}{\lambda \cos \frac{\lambda}{2}} - \frac{\sinh\left(\nu \frac{x}{l}\right)}{\nu \cosh \frac{\nu}{2}} \right) \quad (22)$$

$M_C(t)$ is still given by (10). $M_{C,0}(\omega)$ can now be written

$$M_{C,0}(\omega) = -EI \frac{d^2}{dx^2} U\left(\frac{l}{2}\right) = \frac{F_0 l}{2(\nu^2 + \lambda^2)} \left(\lambda \tan \frac{\lambda}{2} + \nu \tanh \frac{\nu}{2} \right) = \frac{1}{4} F_0 l \cdot D(\lambda, \nu) \quad (23)$$

$$D(\lambda, \nu) = \frac{2}{\nu^2 + \lambda^2} \left(\lambda \tan \frac{\lambda}{2} + \nu \tanh \frac{\nu}{2} \right) \quad (24)$$

Since $\lambda = \lambda\left(\omega, \frac{P}{P_E}\right)$ and $\nu = \nu\left(\omega, \frac{P}{P_E}\right)$ according to (19), (12) is merely a convergent series expansion of the closed form solution (24) for the dynamic amplification factor.

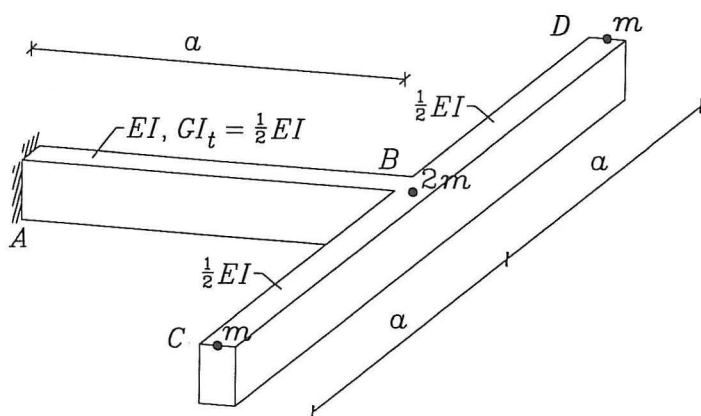
PROBLEM 3

The solution of the problem has been included as example 5-3 in the textbook, and will not be reiterated here.

2.20 June 22, 1998

Duration: 4h

PROBLEM 1



The figure shows a space frame structure consisting of beams AB , BC og BD , which all are placed at same horizontal level. The beams BC and CD are placed end to end along the same line othogonally to beam AB . All beams have the length a , and are assumed to be massless and infinitely stiff against axial deformations. Beam AB has the constant bending stiffness EI against vertical bending deformations and St. Venant torsional stiffness $GI_t = \frac{1}{2}EI$ against axial rotations. The beams BC and BD both have the constant bending stiffness $\frac{1}{2}EI$ against vertical bending deformations. Beam AB is fixed at point A . At the free ends C and D and at the node B point masses of magnitude m , m og $2m$, respectively, are attached. Even though it has not been indicated in the figure it is assumed that the said masses have been supported in such a way that they can only move in the vertical direction. Only small vertical vibrations from the static equilibrium state are considered, and possible warping deformations of beam AB are ignored.

Question 1 (20%. $\mu = 6.7\%$)

Determine the undamped circular eigenfrequencies and eigenmodes of the structure.

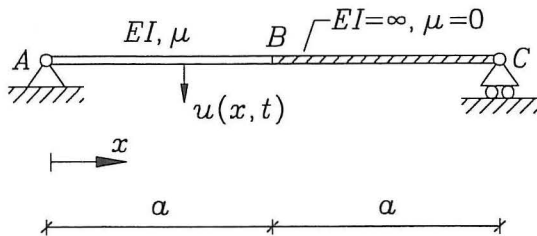
Question 2 (10%. $\mu = 1.1\%$)

Use the symmetry of the structure to formulate equivalent systems of 2 and 1 degrees of freedom for the analysis of symmetric and antisymmetric eigenvibrations, respectively. The mass- and stiffness matrices of the reduced systems are requested.

Help:

$$-x^3 + 20x^2 - 88x + 56 = 0 \text{ has the roots } x = \begin{cases} 3 - \sqrt{5} \\ 3 + \sqrt{5} \\ 14 \end{cases}$$

PROBLEM 2



The figure shows a plane horizontal beam structure consisting of the sub-beams AB and BC , both of the length a . Both beams are assumed to be infinitely stiff against axial deformations, and are free of damping. Beam AB is a Bernoulli-Euler beam with constant bending stiffness EI and constant mass per unit length μ . Beam BC is assumed to be massless ($\mu = 0$) and infinitely stiff against bending deformations ($EI = \infty$). The structure is simply supported at the points A and C . Only small vertical vibrations from the static equilibrium state are considered.

Question 1 (10%. $\mu = 2.6\%$)

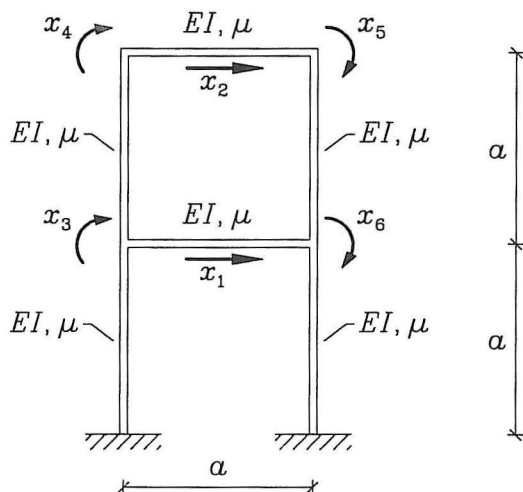
The motion of the elastic sub-beam AB is denoted $u(x, t)$, where x is a coordinate along the beam axis measured from A , and t is the time. Show that the following boundary conditions are valid at the endsection at point B adjacent to the infinitely stiff beam BC

$$u(a, t) = -a \frac{\partial}{\partial x} u(x, t) \Big|_{x=a}, \quad \frac{\partial^2}{\partial x^2} u(x, t) \Big|_{x=a} = -a \frac{\partial^3}{\partial x^3} u(x, t) \Big|_{x=a}$$

Question 2 (20%. $\mu = 10.0\%$)

Formulate the condition for the determination of the undamped circular eigenfrequencies of the considered structural system. Numerical solution is not requested of the frequency condition.

PROBLEM 3



The figure shows a plane 2-storey frame structure. All beams in the structure are Bernoulli-Euler beams with the constant bending stiffness EI and the constant mass per unit length μ . Additionally, all beams are assumed to be infinitely stiff against axial deformations. Both columns in the lower storey are fixed at the ground surface. Only small vibrations in the plane of the structure are considered, and the influence from axial forces on the dynamic response is ignored.

Question 1 (15%. $\mu = 4.6\%$)

Using the global degrees of freedom, show that the global consistent mass matrix and global stiffness matrix of the structure are given as

$$\mathbf{M} = \frac{\mu a}{420} \begin{bmatrix} 1044 & 108 & 0 & -13a & -13a & 0 \\ 108 & 732 & 13a & -22a & -22a & 13a \\ 0 & 13a & 12a^2 & -3a^2 & 0 & -3a^2 \\ -13a & -22a & -3a^2 & 8a^2 & -3a^2 & 0 \\ -13a & -22a & 0 & -3a^2 & 8a^2 & -3a^2 \\ 0 & 13a & -3a^2 & 0 & -3a^2 & 12a^2 \end{bmatrix}$$

$$\mathbf{K} = \frac{EI}{a^3} \begin{bmatrix} 48 & -24 & 0 & 6a & 6a & 0 \\ -24 & 24 & -6a & -6a & -6a & -6a \\ 0 & -6a & 12a^2 & 2a^2 & 0 & 2a^2 \\ 6a & -6a & 2a^2 & 8a^2 & 2a^2 & 0 \\ 6a & -6a & 0 & 2a^2 & 8a^2 & 2a^2 \\ 0 & -6a & 2a^2 & 0 & 2a^2 & 12a^2 \end{bmatrix}$$

The question is considered to be answered if the general principle for the calculation of the matrices is explained, and this is illustrated by working out the calculation of the elements M_{11} and K_{11} .

Question 2 (15%. $\mu = 7.4\%$)

Perform a Guyan system reduction of the described system, using the storey displacements $x_1(t)$ and $x_2(t)$ as master degrees of freedom. Both the mass matrix and the stiffness matrix of the reduced system are requested.

Question 3 (10%. $\mu = 5.0\%$)

Determine an approximate value of the fundamental circular eigenfrequency by use of Rayleigh's fraction, using the static deflection of the structure from a unit horizontal force at the topstorey as shape function.

Help:

The following matrix inverses may be useful at the solution of the problem

$$\frac{EI}{a^3} \begin{bmatrix} 12a^2 & 2a^2 & 0 & 2a^2 \\ 2a^2 & 8a^2 & 2a^2 & 0 \\ 0 & 2a^2 & 8a^2 & 2a^2 \\ 2a^2 & 0 & 2a^2 & 12a^2 \end{bmatrix}^{-1} = \frac{a}{476EI} \begin{bmatrix} 43 & -12 & 5 & -8 \\ -12 & 67 & -18 & 5 \\ 5 & -18 & 67 & -12 \\ -8 & 5 & -12 & 43 \end{bmatrix}$$

$$\mathbf{K}^{-1} = \frac{1}{9240} \frac{a}{EI} \begin{bmatrix} 532a^2 & 700a^2 & 294a & 42a & 42a & 294a \\ 700a^2 & 1610a^2 & 630a & 420a & 420a & 630a \\ 294a & 630a & 1083 & -81 & 249 & 93 \\ 42a & 420a & -81 & 1497 & -153 & 249 \\ 42a & 420a & 249 & -153 & 1497 & -81 \\ 294a & 630a & 93 & 249 & -81 & 1083 \end{bmatrix}$$

SOLUTIONS

PROBLEM 1

Question 1:

The solution of problem 1, question 1 has been included in the textbook as example 3-7, and will not be reiterated here.

Question 2:

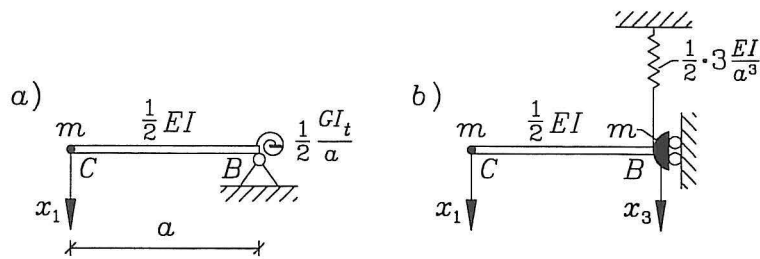


Fig. 1: Equivalent reduced systems. a) Antisymmetric eigenvibrations. b) Symmetric eigenvibrations

The system is geometrically and mechanically symmetric around a plane through beam AB and orthogonal to the plane of the structure. Then the eigenvibrations are either antisymmetric or symmetric around this plane.

In the case of antisymmetric eigenvibrations the point B will not move, and the combined beam CBD must have an inflection point at B . These conditions are equivalent to a simple support at point B . The bending stiffness of the beam AB is not activated, and the torsional stiffness of beam AB of magnitude GI_t/a is shared with one half to each of the beams BC and BD . Hence, the effect of the beam AB on the equivalent system can be equivalized by a rotational spring of the stiffness $\frac{1}{2}GI_t/a$ as shown in fig. 1a. Antisymmetric eigenvibrations may then be analysed by the equivalent system shown in fig. 1a.

The equivalent system has a single degree of freedom $x_1(t)$. The mass is equal to m . The flexibility coefficient becomes

$$\delta_{11} = \frac{a^3}{3 \frac{1}{2} EI} + a \cdot \frac{a}{\frac{1}{2} GI_t} \cdot a = \frac{2}{3} \frac{a^3}{EI} + 2 \frac{a^3}{GI_t} = \frac{14}{3} \frac{a^3}{EI} \quad (1)$$

The spring stiffness of the system becomes

$$k = \frac{1}{\delta_{11}} = \frac{3}{14} \frac{EI}{a^3} \quad (2)$$

The circular eigenfrequency of the antisymmetric (torsional) eigenvibration then follows from (2-7)

$$\omega_1 = \sqrt{\frac{3}{14}} \sqrt{\frac{EI}{ma^3}} \quad (3)$$

In the case of symmetric eigenvibrations, the slope and the shear force of the beam CBD at point B must be zero. This is equivalent to a movable restraint in the midst of the mass $2m$ at point B . The torsional stiffness of the beam AB is not activated, and the bending stiffness of magnitude $3EI/a^3$ is shared with one half to each of the beams BC and BD . Hence, the effect of the beam AB on the equivalent system can be equivalized by a translational spring of stiffness $\frac{1}{2}3EI/a^3$ as shown in fig. 1b. Symmetric eigenvibrations may then be analysed by the equivalent system shown in fig. 1b.

The equivalent system has two degrees of freedom $x_1(t)$ and $x_3(t)$. Only half of the mass at point B is shared by the equivalent system. The mass matrix is then given as

$$\mathbf{M} = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (4)$$

The flexibility coefficients become

$$\delta_{11} = \frac{1}{3} \frac{a^3}{\frac{1}{2}EI} + 2 \frac{a^3}{3EI} = \frac{4}{3} \frac{a^3}{EI} \quad , \quad \delta_{13} = \delta_{31} = \delta_{33} = 2 \frac{a^3}{3EI} = \frac{2}{3} \frac{a^3}{EI} \quad (5)$$

From these the following flexibility- and stiffness matrices are obtained

$$\mathbf{D} = \begin{bmatrix} \delta_{11} & \delta_{13} \\ \delta_{31} & \delta_{33} \end{bmatrix} = \frac{2}{3} \frac{a^3}{EI} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad , \quad \mathbf{K} = \mathbf{D}^{-1} = \frac{3}{2} \frac{EI}{a^3} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \quad (6)$$

The eigenvalue problem (3-42) may be written in the following form

$$\begin{bmatrix} 1 - \lambda_k & -1 \\ -1 & 2 - \lambda_k \end{bmatrix} \begin{bmatrix} \Phi_1^{(k)} \\ \Phi_3^{(k)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad , \quad \lambda_k = \frac{2}{3} \frac{ma^3}{EI} \omega_k^2 \quad , \quad k = 2, 3 \quad (7)$$

The characteristic equation becomes

$$\begin{aligned} \lambda_k^2 - 3\lambda_k + 1 &= 0 \quad \Rightarrow \\ \lambda_k &= \begin{cases} \frac{1}{2}(3 - \sqrt{5}) & , \quad k = 2 \\ \frac{1}{2}(3 + \sqrt{5}) & , \quad k = 3 \end{cases} \Rightarrow \\ \omega_k &= \begin{cases} \sqrt{\frac{3}{4}(3 - \sqrt{5})} \sqrt{\frac{EI}{ma^3}} & , \quad k = 2 \\ \sqrt{\frac{3}{4}(3 + \sqrt{5})} \sqrt{\frac{EI}{ma^3}} & , \quad k = 3 \end{cases} \end{aligned} \quad (8)$$

The eigenmodes are normalized as follows

$$\Phi^{(k)} = \begin{bmatrix} 1 \\ \Phi_3^{(k)} \end{bmatrix} \quad (9)$$

The first equation in (7) is used for the determination of $\Phi_3^{(k)}$

$$\begin{aligned} \Phi_3^{(k)} &= (1 - \lambda_k) \cdot 1 = \frac{1}{2} (\pm\sqrt{5} - 1) \quad \Rightarrow \\ \Phi^{(2)} &= \begin{bmatrix} 1 \\ \frac{1}{2} (\sqrt{5} - 1) \end{bmatrix} \quad , \quad \Phi^{(3)} = \begin{bmatrix} 1 \\ -\frac{1}{2} (\sqrt{5} + 1) \end{bmatrix} \end{aligned} \quad (10)$$

PROBLEM 2

Question 1:

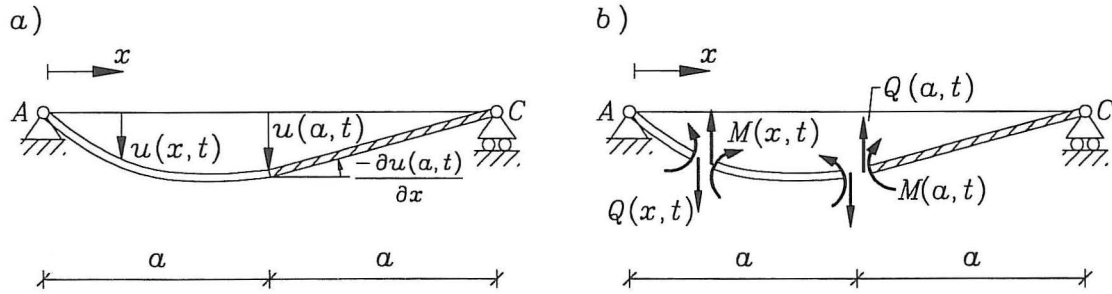


Fig. 1: Boundary conditions at the end-section of the elastic beam AB at point B. a) Geometrical boundary condition. b) Mechanical boundary condition.

With the sign defined in fig. 1a, the rotation of the elastic beam at the boundary section at point B is given as $-\frac{\partial}{\partial x}u(a, t)$. Since the beam BC is infinitely stiff and small deformations have been assumed, this rotation must be equal to $\frac{u(a, t)}{a}$, see fig. 1a. This leads to the following geometrical boundary condition

$$-a \frac{\partial}{\partial x}u(a, t) = u(a, t) \quad (1)$$

The elastic beam AB is cut free from the infinitely stiff beam BC and the shear force $Q(a, t)$, and the bending moment $M(a, t)$ is applied with the sign defined in fig. 1b. Since the beam BC is massless and hence free of inertial loads, these stress resultants must be in equilibrium with the reaction force at point C. Moment equilibrium formulated at point C provides the following relation

$$Q(x, t)a = -M(x, t) \quad (2)$$

The constitutive equation of Bernoulli-Euler beams (4-4) and the statical condition (4-2) gives the conditions

$$\left. \begin{aligned} M(x, t) &= -EI \frac{\partial^2}{\partial x^2}u(x, t) \\ Q(x, t) &= \frac{\partial}{\partial x}M(x, t) = -EI \frac{\partial^3}{\partial x^3}u(x, t) \end{aligned} \right\} \quad (3)$$

Insertion of (3) into (2) then provides the following mechanical boundary condition

$$-a \frac{\partial^3}{\partial x^3}u(a, t) = \frac{\partial^2}{\partial x^2}u(a, t) \quad (4)$$

Question 2:

The eigenvibrations $u(x, t)$ are searched for on the form (4-12)

$$u(x, t) = \Phi(x) \cos(\omega t) \quad (5)$$

If (5) is inserted into the boundary conditions (1) and (4), the following conditions on the amplitude function $\Phi(x)$ are obtained

$$-a \frac{d}{dx} \Phi(a) = \Phi(a) \quad , \quad -a \frac{d^3}{dx^3} \Phi(a) = \frac{d^2}{dx^2} \Phi(a) \quad (6)$$

The differential equation for the amplitude function and the boundary conditions at $x = 0$ are given by (4-13). Introducing the non-dimensional coordinate $\xi = \frac{x}{a}$ the following eigenvalue problem is obtained

$$\left. \begin{aligned} & \frac{d^4}{d\xi^4} \Phi(\xi) + \lambda^4 \Phi(\xi) = 0 \quad , \quad \xi \in]0, 1[\quad , \quad \lambda^4 = \frac{\mu a^4 \omega^2}{EI} \\ & \text{Geometrical boundary conditions:} \\ & \Phi(0) = 0 \quad , \quad \frac{d}{d\xi} \Phi(1) + \Phi(1) = 0 \\ & \text{Mechanical boundary conditions:} \\ & \frac{d^2}{d\xi^2} \Phi(0) = 0 \quad , \quad \frac{d^2}{d\xi^2} \Phi(1) + \frac{d^3}{d\xi^3} \Phi(1) = 0 \end{aligned} \right\} \quad (7)$$

The solution of (7) reads, cf. (4-18)

$$\Phi(\xi) = A \sin(\lambda \xi) + B \cos(\lambda \xi) + C \sinh(\lambda \xi) + D \cosh(\lambda \xi) \quad (8)$$

The boundary conditions $\Phi(0) = \frac{d^2}{d\xi^2} \Phi(0) = 0$ at $\xi = 0$ imply that $B = D = 0$, cf. (4-24). The following solution along with its first three derivatives are then obtained

$$\left. \begin{aligned} & \Phi(\xi) = A \sin(\lambda \xi) + C \sinh(\lambda \xi) \\ & \frac{d}{d\xi} \Phi(\xi) = \lambda (A \cos(\lambda \xi) + C \cosh(\lambda \xi)) \\ & \frac{d^2}{d\xi^2} \Phi(\xi) = \lambda^2 (-A \sin(\lambda \xi) + C \sinh(\lambda \xi)) \\ & \frac{d^3}{d\xi^3} \Phi(\xi) = \lambda^3 (-A \cos(\lambda \xi) + C \cosh(\lambda \xi)) \end{aligned} \right\} \quad (9)$$

Introduction of the boundary conditions at $\xi = 1$ ($x = a$) leads to the following system of homogeneous equations

$$\begin{bmatrix} \sin \lambda + \lambda \cos \lambda & \sinh \lambda + \lambda \cosh \lambda \\ -\sin \lambda - \lambda \cos \lambda & \sinh \lambda + \lambda \cosh \lambda \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (10)$$

Non-trivial solutions are only obtained if the determinant of the coefficient matrix of (10) is zero, which leads to the frequency condition

$$2(\sinh \lambda + \lambda \cosh \lambda)(\sin \lambda + \lambda \cos \lambda) = 0 \Rightarrow$$

$$\sin \lambda + \lambda \cos \lambda = 0 \Rightarrow$$

$$\tan \lambda + \lambda = 0 \quad (11)$$

The first three solutions to (11), as well as the corresponding circular eigenfrequencies ω_j as determined from (4-42), become

$$\lambda_j = \begin{cases} 2.028757 & , \quad j = 1 \\ 4.913180 & , \quad j = 2 \\ 7.978666 & , \quad j = 3 \end{cases} \Rightarrow \quad (12)$$

$$\omega_j = \begin{cases} 4.115855 \sqrt{\frac{EI}{\mu a^4}} & , \quad j = 1 \\ 24.13934 \sqrt{\frac{EI}{\mu a^4}} & , \quad j = 2 \\ 63.65912 \sqrt{\frac{EI}{\mu a^4}} & , \quad j = 3 \end{cases} \quad (13)$$

PROBLEM 3

Question 1:

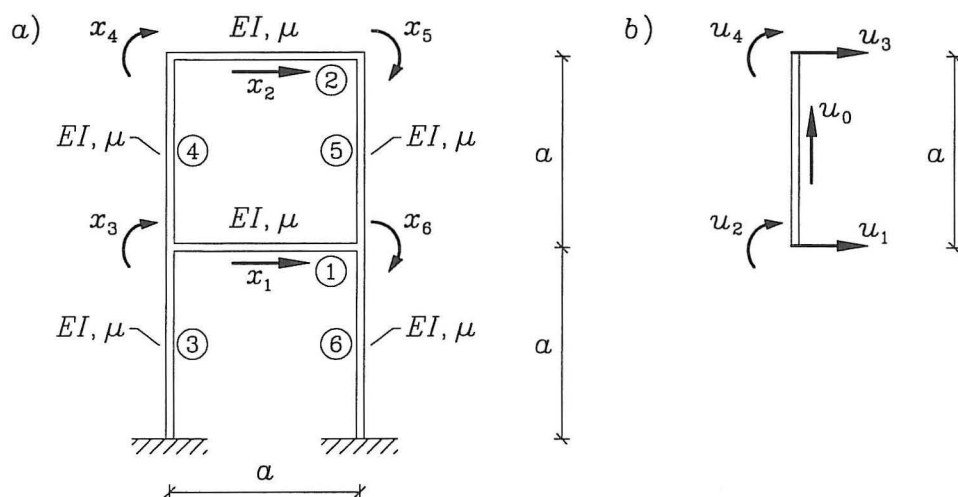


Fig. 1: a) Numbering of beam elements, definition of local coordinate systems and of global degrees of freedom. b) Definition of local degrees of freedom.

Numbering of the beam elements, orientation of the beam elements as well as the definition of the global degrees of freedom are shown in fig. 1a. The definition of the local degrees of freedom is shown in fig. 1b. Each element only has 4 degrees of freedom because the beam element has been assumed to be infinitely stiff against axial deformations. The stiff-body motion of the element in the axial direction is related to the mass μa , which has to be added in the diagonal matrix of the global mass matrix corresponding to the position of the global degrees of freedom x_1 and x_2 . The element mass- and stiffness matrices corresponding to the indicated local degrees of freedom become, cf. (5-42) and (5-43)

$$\mathbf{m}_i = \frac{\mu a}{420} \begin{bmatrix} 156 & 22a & 54 & -13a \\ 22a & 4a^2 & 13a & -3a^2 \\ 54 & 13a & 156 & -22a \\ -13a & -3a^2 & -22a & 4a^2 \end{bmatrix}, \quad \mathbf{k}_i = \frac{EI}{a^3} \begin{bmatrix} 12 & 6a & -12 & 6a \\ 6a & 4a^2 & -6a & 2a^2 \\ -12 & -6a & 12 & -6a \\ 6a & 2a^2 & -6a & 4a^2 \end{bmatrix}, \quad i = 1, \dots, 6(1)$$

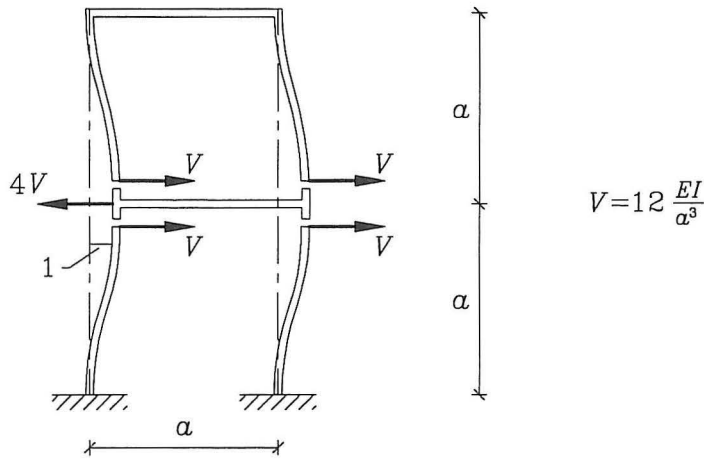


Fig. 2: Restoring force in x_1 -direction from $x_1 = 1$.

Next, the global stiffness matrix is assembled using the standard procedure of the finite element method (or the deformation method) upon summing up the global stiffness contributions at each degree of freedom. For $x_1 = 1$ the restoring force in the same degree of freedom becomes, cf. example 3-2 in the textbook

$$K_{11} = 4 \cdot 12 \frac{EI}{a^3} = 48 \frac{EI}{a^3} \quad (2)$$

The global mass matrix is assembled in the same way as the global stiffness matrix with the components of \mathbf{K} and \mathbf{M} pair-wise related as indicated by the element matrices (1). Additionally, the mass μa needs to be added for the elements M_{11} and M_{22} to account for the inertial load due to the stiff-body motion of the 1st and 2nd storey beams as explained above. M_{11} then becomes

$$M_{11} = 4 \cdot \frac{156}{420} \mu a + \mu a = \frac{1044}{420} \mu a \quad (3)$$

Question 2:

The mass- and stiffness matrices are partitioned into the following form

$$\mathbf{M} = \frac{\mu a}{420} \left[\begin{array}{cc|cccc} 1044 & 108 & 0 & -13a & -13a & 0 \\ 108 & 732 & 13a & -22a & -22a & 13a \\ \hline 0 & 13a & 12a^2 & -3a^2 & 0 & -3a^2 \\ -13a & -22a & -3a^2 & 8a^2 & -3a^2 & 0 \\ -13a & -22a & 0 & -3a^2 & 8a^2 & -3a^2 \\ 0 & 13a & -3a^2 & 0 & -3a^2 & 12a^2 \end{array} \right] = \left[\begin{array}{c|c} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \hline \mathbf{M}_{12}^T & \mathbf{M}_{22} \end{array} \right] \quad (4)$$

$$\mathbf{K} = \frac{EI}{a^3} \left[\begin{array}{cc|cccc} 48 & -24 & 0 & 6a & 6a & 0 \\ -24 & 24 & -6a & -6a & -6a & -6a \\ \hline 0 & -6a & 12a^2 & 2a^2 & 0 & 2a^2 \\ 6a & -6a & 2a^2 & 8a^2 & 2a^2 & 0 \\ 6a & -6a & 0 & 2a^2 & 8a^2 & 2a^2 \\ 0 & -6a & 2a^2 & 0 & 2a^2 & 12a^2 \end{array} \right] = \left[\begin{array}{c|c} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \hline \mathbf{K}_{12}^T & \mathbf{K}_{22} \end{array} \right] \quad (5)$$

The reduced mass matrix then becomes cf. (3-274)

$$\mathbf{M}_{11} = \frac{\mu a}{420} \begin{bmatrix} 1044 & 108 \\ 108 & 732 \end{bmatrix} \quad (6)$$

The reduced stiffness matrix becomes, cf. (3-274)

$$\tilde{\mathbf{K}}_{11} = \mathbf{K}_{11} - \mathbf{K}_{12} \mathbf{K}_{22}^{-1} \mathbf{K}_{12}^T = \frac{42}{119} \frac{EI}{a^3} \begin{bmatrix} 115 & -50 \\ -50 & 38 \end{bmatrix} \quad (7)$$

As a help, the inverse matrix \mathbf{K}_{22}^{-1} has been given subsequent to the problem text.

Question 3:

A unit horizontal force is applied at the top storey (i.e. in the degree of freedom $x_2(t)$). Then $\mathbf{x} = \mathbf{D}\mathbf{f}$ becomes equal to the second column of \mathbf{D} . With the flexibility matrix as given after the problem text, one has

$$\mathbf{f} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} 10a \\ 23a \\ 9 \\ 6 \\ 6 \\ 9 \end{bmatrix} \quad (8)$$

Notice that (8) merely indicates the correct shape of the static deformation rather than the correct magnitude and dimensions from the load \mathbf{f} . Actually, multiplication of the

vector \mathbf{x} by an arbitrary factor does not affect the value of the Rayleigh fraction (3-308). The 2nd column of \mathbf{D} is obtained upon multiplying the indicated vector \mathbf{x} by the factor $\frac{70}{9240} \frac{a^2}{EI}$. The following result is then obtained from (4), (5) and (8)

$$\omega_R^2 = \frac{\mathbf{x}^T \mathbf{K} \mathbf{x}}{\mathbf{x}^T \mathbf{M} \mathbf{x}} = \frac{3036}{\frac{537978}{420}} \frac{EI}{\mu a^4} = 2.370208 \frac{EI}{\mu a^4} \quad (9)$$

The exact lowest eigenvalues based on the complete mass- and stiffness matrices (4) and (5) become

$$\omega_j^2 = \begin{cases} 2.248248 \frac{EI}{\mu a^4} & , \quad j = 1 \\ 24.36476 \frac{EI}{\mu a^4} & , \quad j = 2 \end{cases} \quad (10)$$

The corresponding eigenvalues based on the reduced system mass- and stiffness matrices (6) and (7) become

$$\omega_j^2 = \begin{cases} 2.249407 \frac{EI}{\mu a^4} & , \quad j = 1 \\ 24.27420 \frac{EI}{\mu a^4} & , \quad j = 2 \end{cases} \quad (11)$$

Consequently, the errors introduced by the Guyan reduction are completely ignorable in the present case. Further, the Guyan reduction scheme provides a substantially better estimate for the lowest circular eigenfrequency than does the Rayleigh fraction. Further, results from the considered system have been given in the textbook in examples 3-15 and 3-17.